Jambura Journal of Biomathematics

Check for updates

**Article Identity**
Jambura J. Biomath.
Volume 7 Issue 1 Pages 60 – 74
March 2026, E-ISSN 2723-0317

**Article History**
Received 4 February 2025
Revised 7 July 2026
Accepted 27 January 2026
Published 18 March 2026

**Keywords**
Mathematical analysis, Nitrogen fixation, Sunflower oil, Soil pH

Editorial office: Department of Mathematics, Universitas Negeri Gorontalo, Jln. Prof. Dr. Ing. B. J. Habibie, Bone Bolango 96554, Indonesia

To Cite this Article: Mondal R et al.. Mathematical Modeling and Analysis on Nitrogen Fixation for Controlling Sunflower Oil Production. Jambura Journal of Biomathematics. 2026;7(1):60-74. doi:10.37905/jjbm.v7i1.11

RESEARCH ARTICLE

# Mathematical Modeling and Analysis on Nitrogen Fixation for Controlling Sunflower Oil Production

**Ronjit Mondal**[1], **Jannatul Ferdous Puspo**[1], **Uzzwal Kumar Mallick** [1,✉]

[1]Mathematics Discipline, Khulna University, Khulna-9208, Bangladesh

✉Corresponding author. Email: *mallickuzzwal@math.ku.ac.bd*

**Abstract.** *Nitrogen fixation plays a vital role in enhancing crop yields, yet its inefficient utilization remains a critical challenge in sustainable agriculture. In sunflower oil production, imbalances in nitrogen availability can lead to reduced growth and oil quality, necessitating effective management strategies. To address these challenges, we have developed a newly proposed mathematical model using a system of ordinary differential equations that has been studied both analytically and numerically. Positivity and boundedness of the model's solution, stability at the equilibrium points, and characteristics with respect to state variables have been studied as some parts of the analytical solution. Also, the numerical solution has been done using the Runge–Kutta 4th order technique. The findings of this study reveal that both excessive and insufficient nitrogen application can negatively impact the expected oil production from sunflower crops as well as maintaining soil pH within the standard range, which is essential for rising oil yield. Furthermore, the results indicate that applying nitrogen fertilizer at half the recommended rate between* $30^{th}$ *and* $40^{th}$ *days, followed by the full rate between the* $40^{th}$ *and* $50^{th}$ *days, results in the same expected oil production as applying the full recommended amount evenly throughout the period* $30^{th}$ *to* $50^{th}$ *days. Therefore, this study emphasizes the need for precise nitrogen management methods to improve oil production.*

## 1. Introduction

Nitrogen is essential for the production of sunflower oil because it affects both the quantity and quality of the oil that can be extracted from the seeds. However, the excessive use of synthetic nitrogen fertilizers in agriculture has led to significant environmental problems, including soil degradation, water pollution, and greenhouse gas emissions. This problem highlights the importance of finding more sustainable ways to grow sunflowers, such as using mathematical modeling to improve nitrogen fixation, thereby balancing ecological health and agricultural productivity. These methods can boost oil production while minimizing the harm to the environment, which will be good for both farmers and ecosystems in the long run [1, 2].

Sunflower (Helianthus annuus), a flowering plant in the Asteraceae family, is grown worldwide for its oil-rich seeds. Although its primary purpose is to produce oil, it is also prized for its beauty and edible seeds. Originally native to the Americas, sunflowers are now grown in temperate climates around the world, including Europe, Asia, and Africa [3, 4]. In addition to their beneficial oil content, sunflower seeds are incredibly nutrient-dense, providing a wealth of proteins, fibre, vitamins, and minerals [5, 6]. Notably, sunflower oil is recognized for its high concentration of polyunsaturated fatty acids and vitamin E, both of which are linked to various health benefits that positively contribute to overall well-being. In fact, research has shown that sunflower oil has the potential to lower cholesterol levels, reduce the risk of cardiovascular diseases [7, 8]. Furthermore, recent market analyses indicate that the global sunflower oil market continues to expand steadily due to increasing demand for healthier vegetable oils and plant-based products [9].

Although sunflower seeds are an excellent source of nutrients, the nutritional value of sunflower crops is often affected by the use of excessive nitrogen fertilisers and pest infestations. These factors not only lead to reduced yield and poor oil quality but also cause significant environmental harm [10, 11]. In particular, pests such as the sunflower seed weevil can significantly damage sunflower crops, affecting both seed quality and the oil extraction process [12, 13]. Although many studies have focused on various pest control methods, including the use of chemical pesticides, these approaches have raised concerns about their negative impact on both the environment and human health [14, 15]. Consequently, there has been growing interest in exploring alternative, sustainable pest control methods, such as biological control, which aim to reduce reliance on chemical pesticides while preserving crop yields and improving oil quality [16, 17].

Several studies have explored the effectiveness of biological control in sunflower cultivation. For example, natural predators and parasitoids have been shown to suppress pest populations without causing environmental harm [12]. Additionally, plant-based biopesticides and ecological pest-management techniques can reduce pest infestation while improving sunflower yield and oil content [16, 18]. Mathematical modeling approaches based on predator–prey interactions have also been used to analyse the dynamics between pests and their natural enemies, helping to support the development of integrated pest management strategies for sunflower cultivation. One widely used framework for describing such biological interactions is the Lotka–Volterra equations, which provides a theoretical basis for studying pest–predator population dynamics in agricultural ecosystems.

However, despite these advances, few studies have explored the role of nitrogen fixation in optimising sunflower oil production, primarily through the use of mathematical models. There is also limited research on integrating sustainable nitrogen management to achieve higher yields and better oil quality. This gap in the literature motivated our study, which uses a system of ordinary differential equations (ODEs) to model nitrogen fixation and its effects on sunflower growth and oil yield. By optimising nitrogen use while not considering the impacts of pest control, disease and soil dynamics, our model aims to maximise oil production and improve the sustainability of sunflower farming.

This study contributes to the field by developing a mathematical model that combines nitrogen fixation and crop yield optimisation. Through this innovative approach, we aim to reduce the reliance on chemical fertilisers and pesticides while increasing sunflower yield and oil quality. Therefore, this research not only addresses gaps in the existing literature but also provides valuable insights into sustainable farming practices that can benefit both farmers and the environment.

The rest of the paper is organised as follows: Section 2 illustrates the model formulation with the help of the system of ODE, where Section 3 demonstrates the different types of analytical behaviour of the model. After that, the numerical simulation results and discussion are presented in Section 4. Finally, Section 5 presents the main conclusions and discusses their implications.

## 2. Model Formulation

In this part of the article, we aim to develop a newly proposed mathematical model based on agricultural literature, including [19–21], to analyze nitrogen fixation and its role in sunflower cultivation. Our objective is to improve sunflower seed and oil output by optimizing nutrient management while taking into account the unique difficulties faced by Bangladeshi agriculture.

In developing our model, we focus on three key state variables: $Y(t)$, which represents the sunflower oil production; $N(t)$, which denotes the nitrogen concentration in the soil; and $pH(t)$, which indicates the soil pH. These state variables interact dynamically to capture the influence of soil conditions and nutrient availability on sunflower growth and oil yield. Thus, our mathematical model integrates these interactions to describe the system's behavior over time. The model has limitations, even though it sheds light on nitrogen fixation and oil output. It assumes that nitrogen fixation is the sole source of nitrogen fertilizer, while also ignoring biological effects such as pests and diseases. Additionally, the model ignores soil dynamics, particularly in terms of microbial interactions and nutrient cycling, which can affect the accuracy of the model under certain real-world conditions.

The production of sunflower oil $Y(t)$, is directly impacted by the levels of soil nitrogen $N(t)$ and the pH of the soil $pH(t)$. In the absence of nitrogen fixation, soil nitrogen levels decrease due to plant uptake and natural leaching processes. However, nitrogen fixation by soil bacteria can replenish nitrogen levels, thus improving sunflower growth and oil yield. Furthermore, soil pH plays a crucial role by affecting both nutrient availability and microbial activity. An imbalance in pH can hinder nitrogen uptake, thus limiting the production of sunflower oil. In contrast, excessive use of chemical fertilizers can alter the soil pH unfavorably, while organic fertilizers and microbial inoculants can improve nitrogen fixation and stabilize soil pH, enhancing sunflower oil yield. Taking into account these factors, the first differential equation will be formulated as follows.

$$\frac{dY}{dt} = \alpha Y + \beta(pH - pH_0)(pH_f - pH)Y + \gamma(N - N_0)(N_f - N)Y - \delta Y, \tag{1}$$

where $\alpha$ represents the rate at which photosynthesis contributes to sunflower oil production, and $\delta$ is the rate at which sunflower oil production decreases due to soil salinity. $\beta$ represents the rate at which soil pH impacts oil production, and $\gamma$ denotes the effect of nitrogen availability on oil production.

After this, the presence of nitrogen inside the soil is a significant factor contributing to its productivity. The application of organic fertilizer provides the potential to increase nitrogen levels in the soil. On the other hand, soil nitrogen declines as plants utilize it for crop growth and yield. The entirety of this represents a mathematical equation,

$$\frac{dN}{dt} = \mu N - \psi NY, \tag{2}$$

where $\mu$ is the rate at which nitrogen concentration increases due to organic fertilizer application, whereas $\psi$ is the rate at which nitrogen is depleted due to sunflower growth and oil production.

Lastly, the soil pH is influenced by both organic and chemical fertilizers. The application of organic fertilizers typically increases soil pH, whereas chemical fertilizers such as gypsum and potassium sulfate tend to lower it. So, our last equation will be

$$\frac{dpH}{dt} = \mu_1 - \phi pH, \tag{3}$$

where $\mu_1$ is the rate at which organic fertilizers contribute to an increase in soil pH, and $\phi$ is the rate at which soil pH decreases due to chemical fertilizers. Finally, our model equations will be,

$$\begin{aligned} f_1(Y,N,pH) &= \frac{dY}{dt} = \alpha Y + \beta(pH - pH_0)(pH_f - pH)Y + \gamma(N - N_0)(N_f - N)Y - \delta Y, \\ f_2(Y,N,pH) &= \frac{dN}{dt} = \mu N - \psi NY, \\ f_3(Y,N,pH) &= \frac{dpH}{dt} = \mu_1 - \phi pH, \end{aligned} \tag{4}$$

with the initial conditions $Y(0) = Y_0(t) \geq 0, N(0) = N_0 \geq 0$ and $pH(0) = pH_7 \geq 0$.

## 3. Analytical Analysis of the Model

### *3.1. Positivity and boundedness*

**Lemma 1.** *Considering $Y(0) \geq 0, N(0) \geq 0$ and $pH(0) \geq 0$, then $Y(t)$, $N(t)$ and $pH(t)$ will be always positive for all $t \in [0,T]$ in $R_3^+$ where $T \geq 0$.*

*Proof.* Taking all parameters of the system and all initial values to be positive, we have to prove that $Y(t)$, $N(t)$ and $pH(t)$ will be positive for all $t \in [0,T]$ in $R_3^+$. From the eq. (3), we can write as follows:

$$\begin{aligned} \frac{dpH}{dt} &= \mu_1 - \phi pH, \\ \frac{dpH}{dt} &\geq -\phi pH, \\ \frac{dpH}{pH} &\geq -\phi dt. \end{aligned}$$

Integrating both sides we get,

$$\ln pH - \ln pH_7 \geq -\phi t.$$

Then this result will show as

$$\begin{aligned} \ln \frac{pH}{pH_7} &\geq -\phi t, \\ \frac{pH}{pH_7} &\geq e^{-\phi t}, \\ pH(t) &\geq pH_7 e^{-\phi t} > 0. \end{aligned}$$

Again, from eq. (2), we can write

$$\begin{aligned} \frac{dN}{dt} &= \mu N - \psi NY, \\ \frac{dN}{dt} &= (\mu - \psi Y)N, \\ \frac{dN}{N} &= (\mu - \psi Y)dt. \end{aligned}$$

Integrating both sides we get,

$$N(t) = N_0 e^{\mu t - \psi \int Y dt} > 0.$$

Since we have already found that $N > 0, pH > 0$ as well as

$$(pH - pH_0)(pH_f - pH) = p_1 > 0, \quad (N - N_0)(N_f - N_0) = n_1 > 0$$

Then from eq. (1)

$$\begin{aligned} \frac{dY}{dt} &= \alpha Y + \beta(pH - pH_0)(pH_f - pH)Y + \gamma(N - N_0)(N_f - N)Y - \delta Y, \\ \frac{dY}{dt} &= (\alpha + \beta p_1 + \gamma n_1 - \delta)Y, \\ \frac{dY}{Y} &= (\alpha + \beta p_1 + \gamma n_1 - \delta)dt. \end{aligned}$$

Integrating both sides, we get,

$$\begin{aligned} \ln Y(t) &= (\alpha + \beta p_1 + \gamma n_1 - \delta)t, \\ Y(t) &= e^{(\alpha + \beta p_1 + \gamma n_1 - \delta)t} > 0. \end{aligned}$$

These all complete the proof. ■

**Lemma 2.** *The functions $f_1$, $f_2$ and $f_3$ are bounded within a closed interval $[a,b]$.*

*Proof.* We know that a function is said to be continuous on a closed interval $[a,b]$ when the function is defined at every point on that interval $[a,b]$ and undergoes no interruptions, jumps or breaks.

From eq. (4), it is clear that all the functions $f_1$, $f_2$ and $f_3$ are not undefined for any point. So, they must be defined for every point in an interval $[a,b]$ as we perform for prediction by using this model for 30-50 days. So for our model's, $a = 30$ and $b = 50$. So, we can easily say that the functions $f_1$, $f_2$ and $f_3$ are continuous within the closed interval or domain $[a,b]$

According to the boundedness theorem [22], a continuous function on a closed interval must be bounded on that interval. So, our considering functions will also be bounded in the interval $[a,b]$. This completes our proof. ■

### 3.2. Equilibrium analysis

By solving the following equation, we can obtain the equilibrium point of the model,

$$0 = \alpha Y + \beta(pH - pH_0)(pH_f - pH)Y + \gamma(N - N_0)(N_f - N)Y - \delta Y, \tag{5}$$

$$0 = \mu N - \psi NY, \tag{6}$$

$$0 = \mu_1 - \phi pH. \tag{7}$$

Solving the eqs. (5) to (7), we get two non-negative equilibrium points $E_1$ and $E_2$, where

$$E_1 = \left(0, 0, \frac{\mu_1}{\phi}\right),$$

$$E_2 = \left(\frac{\mu}{\psi}, \frac{\sqrt{\frac{\varepsilon+\varepsilon_1+\tau-\tau_1}{\gamma}}}{2\phi}, \frac{\mu_1}{\phi}\right),$$

where,

$$\begin{aligned} N_0\phi + N_f\phi &= \varepsilon, \\ 4(\alpha\phi^2 + \delta\phi^2 + \beta\mu_1 pH_f\phi - \beta\mu_1^2 - \beta pH_0 pH_f\phi^2) &= \varepsilon_1, \\ \gamma N_0^2\phi^2 + \gamma N_f^2\phi^2 &= \tau, \\ 2\gamma N_0 N_f\phi^2 &= \tau_1, \end{aligned}$$

which are positive under the condition $\varepsilon + \varepsilon_1 + \tau > \tau_1$.

### 3.3. Status of expected sunflower oil

Using the next generation matrix method mentioned in [23], we will find the status of expected sunflower oil $R_0$. To examine the dynamics of expected sunflower oil, this state variable is associated with the discussion. We take

$$\frac{dY}{dt} = \alpha Y + \beta(pH - pH_0)(pH_f - pH)Y + \gamma(N - N_0)(N_f - N)Y - \delta Y.$$

Differentiating it with respect to $Y$, we get

$$\frac{d}{dY}\left(\frac{dY}{dt}\right) = \alpha + \beta pH(pH_f - pH) - \beta pH_0(pH_f - pH) + \gamma N(N_f - N) - \gamma N_0(N_f - N) - \delta,$$

$$\frac{d}{dY}\left(\frac{dY}{dt}\right) = [\alpha + \beta pH(pH_f - pH) + \gamma N(N_f - N)] - [\beta pH_0(pH_f - pH) - \gamma N_0(N_f - N) - \delta].$$

Therefore, at the equilibrium point $(Y^*, N^*, pH^*)$ two matrices $M$ and $D$ which represent increase and decrease of expected sunflower soil are $M = \alpha + \beta pH(pH_f - pH) + \gamma N(N_f - N)$ and $D = \beta pH_0(pH_f - pH) - \gamma N_0(N_f - N) - \delta$. Then the status of expected sunflower oil $R_0$ will be

$$R_0 = \frac{M}{D} = \frac{\alpha + \beta pH(pH_f - pH) + \gamma N(N_f - N)}{\beta pH_0(pH_f - pH) - \gamma N_0(N_f - N) - \delta}. \tag{8}$$

Consequently, excepted sunflower oil will be raised if

$$\alpha + \beta pH(pH_f - pH) + \gamma N(N_f - N) > \beta pH_0(pH_f - pH) - \gamma N_0(N_f - N) - \delta.$$

In contrast, excepted sunflower oil will be decreased if

$$\alpha + \beta pH(pH_f - pH) + \gamma N(N_f - N) < \beta pH_0(pH_f - pH) - \gamma N_0(N_f - N) - \delta.$$

### 3.4. Stability analysis of the model

**local stability**

**Theorem 3.** *(Local Stability Theorem) The dynamical system is unstable at equilibrium point $E_1$.*

*Proof.* Firstly, we consider

$$f_1(Y, N, pH) = \alpha Y + \beta(pH - pH_0)(pH_f - pH)Y + \gamma(N - N_0)(N_f - N)Y - \delta Y = 0, \tag{9}$$
$$f_2(Y, N, pH) = \mu N - \psi NY = 0, \tag{10}$$
$$f_3(Y, N, pH) = \mu_1 - \phi pH = 0. \tag{11}$$

For the eqs. (9) to (11), the Jacobian matrix is

$$J = \frac{\partial(f_1, f_2, f_3)}{\partial(Y, N, pH)} = \begin{pmatrix} \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial pH} \\ \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial pH} \\ \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial N} & \frac{\partial f_3}{\partial pH} \end{pmatrix},$$

$$J = \begin{pmatrix} c + \beta(pHpH_f + pH^2 - pH_0 pH) + \gamma(NN_f + N^2 - N_0 N) & \gamma(b - 2N)Y & \beta(u - 2pH)Y \\ -\psi N & \mu - \psi Y & 0 \\ 0 & 0 & -\phi \end{pmatrix}, \tag{12}$$

where, $c = \alpha - \beta pH_0 pH_f - \gamma N_0 - \delta$, $b = N_f + N_0$ and $u = pH_f - pH_0$. At the equilibrium point $E_1 = (0, 0, \frac{\mu_1}{\phi})$, the eq. (12) becomes

$$J_{E_1} = \begin{pmatrix} c + \beta(\frac{\mu_1}{\phi} pH_f + (\frac{\mu_1}{\phi})^2 - pH_0 \frac{\mu_1}{\phi}) & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\phi \end{pmatrix}.$$

So, the characteristics equation is

$$|J_{E_1} - \lambda I| = 0,$$

$$\begin{vmatrix} c + \beta(\frac{\mu_1}{\phi} pH_f + (\frac{\mu_1}{\phi})^2 - pH_0 \frac{\mu_1}{\phi}) - \lambda & 0 & 0 \\ 0 & \mu - \lambda & 0 \\ 0 & 0 & -\phi - \lambda \end{vmatrix} = 0,$$

$$(-\lambda-\phi)\begin{vmatrix} c+\beta(\tfrac{\mu_1}{\phi}pH_f+(\tfrac{\mu_1}{\phi})^2-pH_0\tfrac{\mu_1}{\phi})-\lambda & 0 \\ 0 & \mu-\lambda \end{vmatrix}=0,$$
$$(-\lambda-\phi)(c+\beta(\frac{\mu_1}{\phi}pH_f+(\frac{\mu_1}{\phi})^2-pH_0\frac{\mu_1}{\phi})-\lambda)(\mu-\lambda)=0.$$

From this equation, it is clear that there are three different values of eigenvalue ($\lambda$) and they are

$$\lambda_1=-\phi,$$
$$\lambda_2=c+\beta\left(\frac{\mu_1}{\phi}pH_f+(\frac{\mu_1}{\phi})^2-pH_0\frac{\mu_1}{\phi}\right),$$
$$\lambda_3=\mu.$$

Thus, we see that all the eigenvalues are not negative, so our model is unstable at the equilibrium point $E_1$. ■

**Theorem 4.** *(Local Stability Theorem) the equilibrium point $E_2$ is asymptotically stable provided conditions $\psi e_2\gamma(b-2e_2)\eta<0$ and unstable otherwise.*

*Proof.* At the equilibrium point

$$E_2=\left(\frac{\mu}{\psi},\frac{\sqrt{\frac{\varepsilon+\varepsilon_1+\tau-\tau_1}{\gamma}}}{2\phi},\frac{\mu_1}{\phi}\right).$$

Equation (12) becomes,

$$J_{E_2}=\begin{pmatrix} c+\beta_1+\gamma_1 & \gamma(b-2e_2).\eta & \beta(u-2\eta_1).\eta \\ -\psi e_2 & 0 & 0 \\ 0 & 0 & -\phi \end{pmatrix},$$

where,

$$\beta_1=\left(\frac{\mu_1}{\phi}pH_f+\left(\frac{\mu_1}{\phi}\right)^2-pH_0\frac{\mu_1}{\phi}\right),\quad \gamma_1=e_2N_f+e_2^2-e_2N_0,\quad u=pH_f-pH_0,$$
$$\eta=\frac{\mu}{\psi},\quad \eta_1=\frac{\mu_1}{\phi},\quad e_2=\frac{\sqrt{\frac{\varepsilon+\varepsilon_1+\tau-\tau_1}{\gamma}}}{2\phi}$$

So, the characteristics equation is

$$|J_{E_2}-\lambda I|=0,$$
$$\begin{vmatrix} c+\beta_1+\gamma_1-\lambda & \gamma(b-2e_2).\eta & \beta(u-2\eta_1).\eta \\ -\psi e_2 & -\lambda & 0 \\ 0 & 0 & -\phi-\lambda \end{vmatrix}=0,$$
$$(-\lambda-\phi)(-(c+\beta_1+\gamma_1-\lambda)\lambda+\psi e_2\gamma(b-2e_2)\eta)=0.$$

Then we get,

$$\lambda_1=-\phi$$

which is negative. And

$$\lambda_2=\frac{(c+\beta_1+\gamma_1)\pm\sqrt{(c+\beta_1+\gamma_1)^2-4\cdot1\cdot\psi e_2\gamma(b-2e_2)\eta}}{2}. \tag{13}$$

The value of $\lambda_2$ will be negative under conditions $\psi e_2\gamma(b-2e_2)\eta<0$. This implies that the equilibrium point $E_2$ is asymptotically stable and otherwise unstable. ■

### 3.5. Characteristics of states for equilibrium values with respect to $\mu$

We will discuss the characterization of the equilibrium values of expected oil production, nitrogen, and pH with respect to $\mu$. From eqs. (1) to (3) we obtain two functions of $Y^*$, $N^*$ and $\mu$ in below:

$$\begin{aligned} f(Y^*,N^*,\mu) &= (\alpha+\beta mm_1-\delta)Y+\gamma(N-N_0)(N_f-N)Y,\\ g(Y^*,N^*,\mu) &= \mu N-\psi NY, \end{aligned}$$

where, $m=\frac{\mu_1}{\phi}-pH_0$ and $m_1=pH_f-\frac{\mu_1}{\phi}$

$$\begin{aligned} \frac{dY^*}{d\mu} &= \frac{\begin{vmatrix} \frac{\partial f(Y^*,N^*,\mu)}{\partial N^*} & \frac{\partial f(Y^*,N^*,\mu)}{\partial \mu} \\ \frac{\partial g(Y^*,N^*,\mu)}{\partial N^*} & \frac{\partial g(Y^*,N^*,\mu)}{\partial \mu} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f(Y^*,N^*,\mu)}{\partial Y^*} & \frac{\partial f(Y^*,N^*,\mu)}{\partial N^*} \\ \frac{\partial g(Y^*,N^*,\mu)}{\partial Y^*} & \frac{\partial g(Y^*,N^*,\mu)}{\partial N^*} \end{vmatrix}} = \frac{\frac{\partial f}{\partial N}\frac{\partial g}{\partial \mu}-\frac{\partial f}{\partial \mu}\frac{\partial g}{\partial N}}{\frac{\partial f}{\partial Y}\frac{\partial g}{\partial N}-\frac{\partial f}{\partial N}\frac{\partial g}{\partial Y}}, \\ &= \frac{\gamma(N_f-2N^*+N_0)N^*}{\alpha+\beta mm'-\delta+\gamma(N^*-N_0)(N_f-N^*)+\gamma\psi N^*(N_f-2N^*+N_0)}. \end{aligned}$$

It is clear that the numerator is positive under the condition $N_f+N_0>N^*$ and under the conditions $\alpha+\beta mm'>\delta$ and $N^*>N_0$ the denominator is also positive. $\therefore \frac{dY^*}{d\mu}>0$ which shows that $Y^*$ is increasing when $\mu$ is increasing. Again,

$$\begin{aligned} \therefore \frac{dN^*}{d\mu} &= \frac{\begin{vmatrix} \frac{\partial f(Y^*,N^*,\mu)}{\partial \mu} & \frac{\partial f(Y^*,N^*,\mu)}{\partial Y^*} \\ \frac{\partial g(Y^*,N^*,\mu)}{\partial \mu} & \frac{\partial g(Y^*,N^*,\mu)}{\partial Y^*} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f(Y^*,N^*,\mu)}{\partial Y^*} & \frac{\partial f(Y^*,N^*,\mu)}{\partial N^*} \\ \frac{\partial g(Y^*,N^*,\mu)}{\partial Y^*} & \frac{\partial g(Y^*,N^*,\mu)}{\partial N^*} \end{vmatrix}} = \frac{\frac{\partial f}{\partial \mu}\frac{\partial g}{\partial Y}-\frac{\partial f}{\partial Y}\frac{\partial g}{\partial \mu}}{\frac{\partial f}{\partial Y}\frac{\partial g}{\partial N}-\frac{\partial f}{\partial N}\frac{\partial g}{\partial Y}}, \\ &= \frac{[\alpha+\beta mm'-\delta+\gamma(N^*-N_0)(N_f-N^*)N^*]}{\alpha+\beta mm'-\delta+\gamma(N^*-N_0)(N_f-N^*)+\gamma\psi N^*(N_f-2N^*+N_0)}. \end{aligned}$$

Since, the numerator is positive and denominator is also positive under the previous condition. $\therefore$ $\frac{dN^*}{d\mu}>0$, which indicates that $N^*$ is increasing when $\mu$ is increasing.

## 4. Numerical Simulations and Discussion

To analyze our newly proposed mathematical model, we have used MATLAB (R2018a). The fourth-order Runge-Kutta method and the parameter values obtained from Table 1 were both used to solve the system. In order to solve the model numerically, we used a number of different parameter values. Our research is related to the following table, which shows a set of logical parameter values taken from Table 1. This set of values was found in a collection of data [24–28]. As an additional point of interest, we have assumed several parameter values by utilizing this collection. In our country, it is predicted that the yield of sunflower per hectare varies between 1.5 and 1.8 tonnes, depending on the nitrogen level per hectare, which ranges from 0.09 to 0.12 tonnes, and the pH level, which lies between 6 and 7.5. Therefore, we have taken into consideration the initial values of the variables for our model analysis, which are $Y(0)=1.6$ Ton/Hectare, $N(0)=0.12$ Ton/Hectare, and $pH(0)=7.5$, respectively.

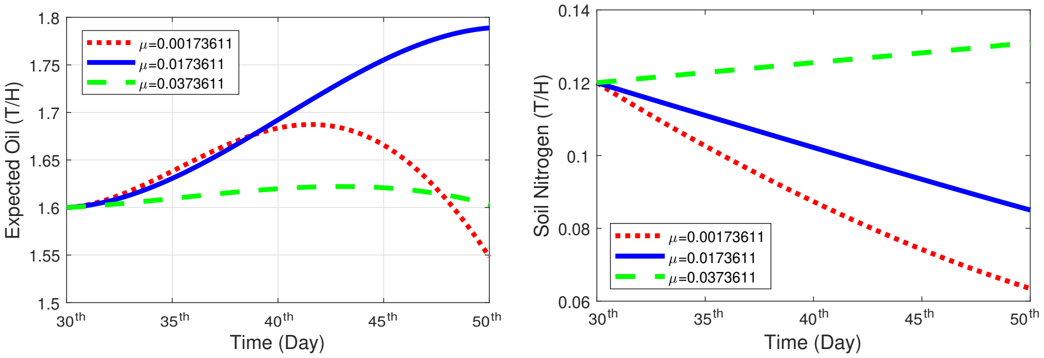
### 4.1. Parametric Variation of the State Variables

Figure 1 shows how state variables change with $\mu$, the rate at which organic fertilizer boosts soil nitrogen. While most plants tend to benefit from increased fertilizer application, sunflowers behave

**Table 1.** Description of parameters with values

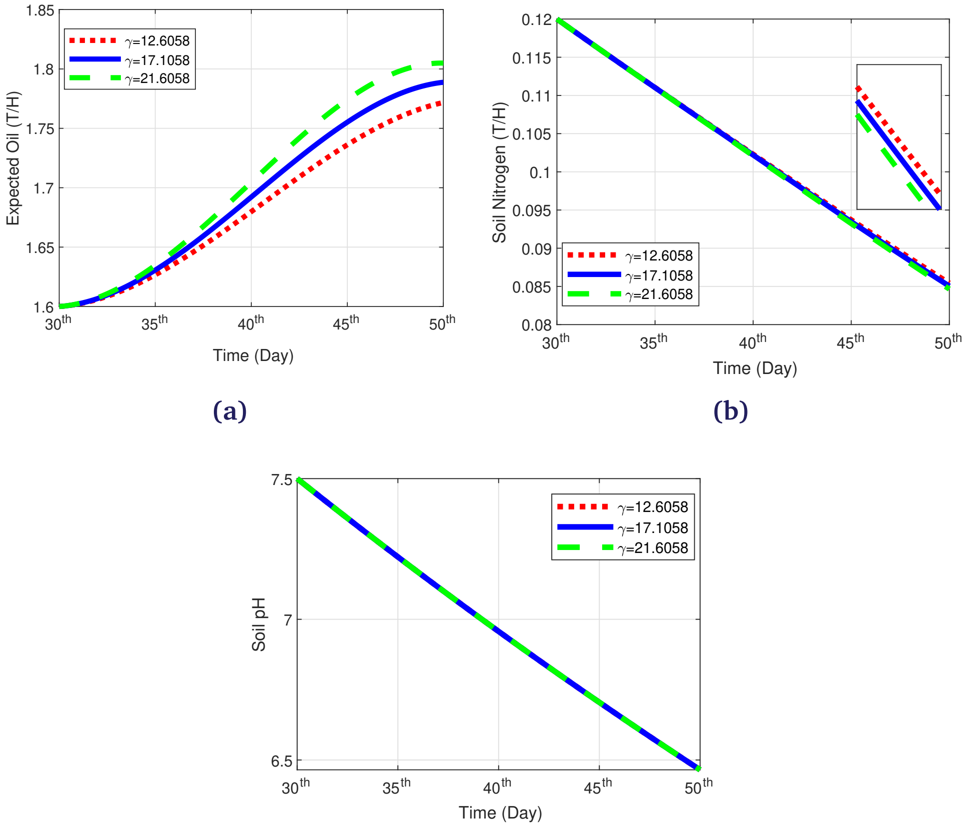
| Parameter | Value | Descriptions of the parameters | Source |
|---|---|---|---|
| $\alpha$ | 0.00068423 | increasing rate of sunflower oil for photosynthesis | Estimated |
| $\beta$ | $6.84225 \times 10^{-}3$ | rising rate of sunflower oil due to pH balance | [26] |
| $\gamma$ | $17.1058\ H^2/T^2$ | increasing rate of sunflower oil due to Nitrogen balance | [26, 27] |
| $\delta$ | 0.000001021242 | reducing rate of sunflower oil due to soil salinity | [27] |
| $\psi$ | $0.020425\ H/T$ | nitrogen conversion rate into sunflower oil | [26] |
| $\mu$ | 0.0173611 | increasing rate of nitrogen due to organic fertilizer | Estimated |
| $\phi$ | 0.009921 | decreasing rate of pH due to chemical fertilizer | Estimated |
| $\mu_1$ | 0.0173611 | increasing rate of pH due to organic fertilizer | Estimated |

differently. Excessive nitrogen in the soil hinders the achievement of optimal yield. Figure 1 has two different figures showing the changes in expected oil production and the change in soil nitrogen, respectively, due to different amounts of organic fertilizer. Figure 1a indicates that the sunflower oil production reaches its peak when soil nitrogen levels remain within the standard range. Again, when the nitrogen level exceeds the standard value, there is a consistent decrease in the expected sunflower oil production in comparison to our predicted oil production. Additionally, the drop in nitrogen levels in the soil leads to a decrease in oil production. Figure 1b illustrates the variation over time of nitrogen concentration in the soil. There is an overall decline in nitrogen content with random minor increases. This occurs due to the sunflower plants' ability to absorb nitrogen from the soil during their growth. Subsequently, a decrease in the quantity of organic fertilizers results in a corresponding reduction in the nitrogen content of the soil. Figure 1b additionally demonstrates that the utilization of organic fertilizers marginally enhances the nitrogen content in the soil. Since, $\mu$ only appears in the equation governing the state variables $Y$ and $N$, it does not directly affect the dynamics of $pH$. As a result, variations in $\mu$ have no influence on the evolution of $pH$. As we know, $\gamma$ refers to the increasing rate of sunflower oil due to the Nitrogen balance. So the changes in $\gamma$ have some effects on the soil nitrogen amount as well as the expected oil production, but it does not affect the soil pH as shown in Figure 2c because there is no relation to pH with $\gamma$. It can be seen from Figure 2a when $\gamma$ rises between the 30th and 45th day, the predicted oil production simultaneously rises; however, after the 45th day, sunflower oil production starts to decline dramatically. This happens because producing sunflower oil is hindered by excessive nitrogen. Again, the variation of $\gamma$ results in a slight change in soil nitrogen, as shown in Figure 2b. An increase in $\gamma$ causes oil output to rise, and nitrogen is



**(a)** **(b)**

**Figure 1.** (a)The predicted oil production is subject to variation depending on the increase in the use of organic fertilizer. (b) Increased use of organic fertilizer to enhance nitrogen content in soil.



**(a)** **(b)**

**(c)**

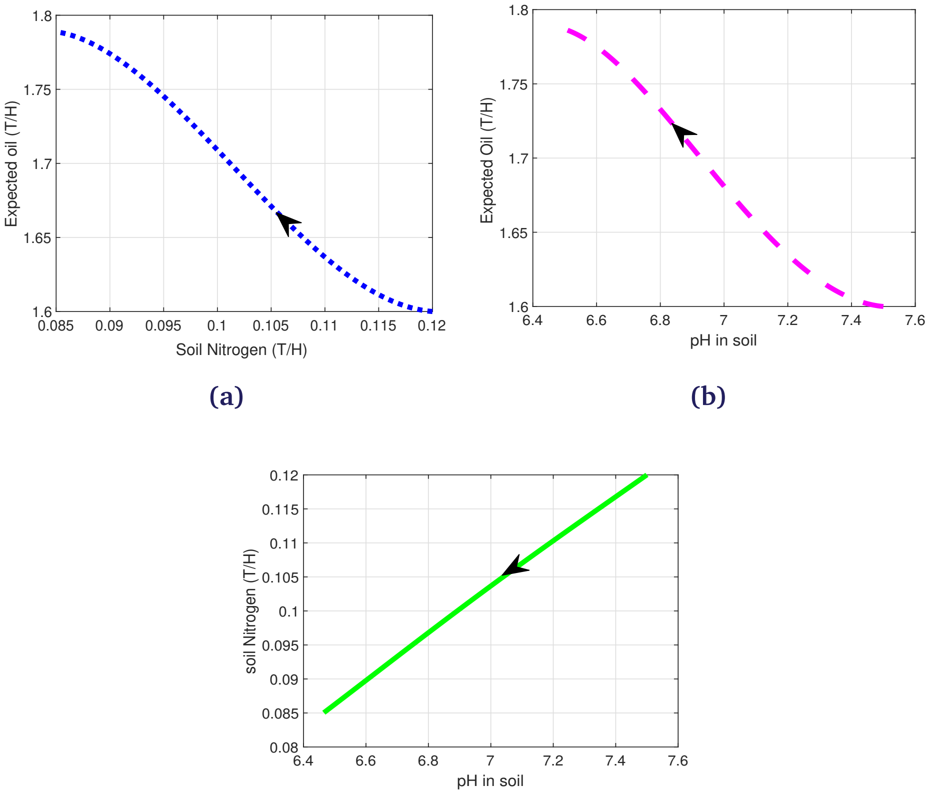
**Figure 2.** (a) Sunflower oil production has increased as a result of the rising nitrogen rate. (b) Because of the production of sunflower oil, the nitrogen content of the soil falls. (c) There was no change in the pH of the soil as a result of the rate having no relation to it.

needed for oil production. As a result, the nitrogen level eventually drops. In other words, a lower $\gamma$ corresponds to a lower oil production Figure 2b. Consequently, there is more wasted nitrogen in the soil, which raises the soil nitrogen level. Again, very high $\gamma$ indicates high nitrogen consumption and oil production; in that scenario, the amount of nitrogen in the soil decreases slightly Figure 2b.

### *4.2. Phase Portrait*

In this section, we will use the phase portrait obtained from simulations of our model to explain its qualitative behaviour. The relationship between soil nitrogen and sunflower oil is shown in Figure 3a.

As the nitrogen level in the soil approached and then remained at the recommended standard, the production of sunflower oil initially increased, demonstrating the positive impact of nitrogen on the crop's growth and seed oil content. However, once the optimal nitrogen level was reached, further increases provided no additional benefit, indicating that a balanced and appropriate nitrogen application is crucial for maximizing sunflower oil yields. Figure 3b suggests a direct correlation between maintaining a specific pH level and maximizing sunflower oil yield. By keeping the pH within the standard range, the oil production increases. Conversely, any deviation from this standard pH results in a significant loss of oil. This implies that monitoring and maintaining the pH level is crucial for optimizing sunflower oil production and minimizing waste. Figure 3c shows that as the soil's pH reaches a standard level, the availability of nitrogen, a crucial nutrient for plant growth, increases. This occurs



(a) (b)

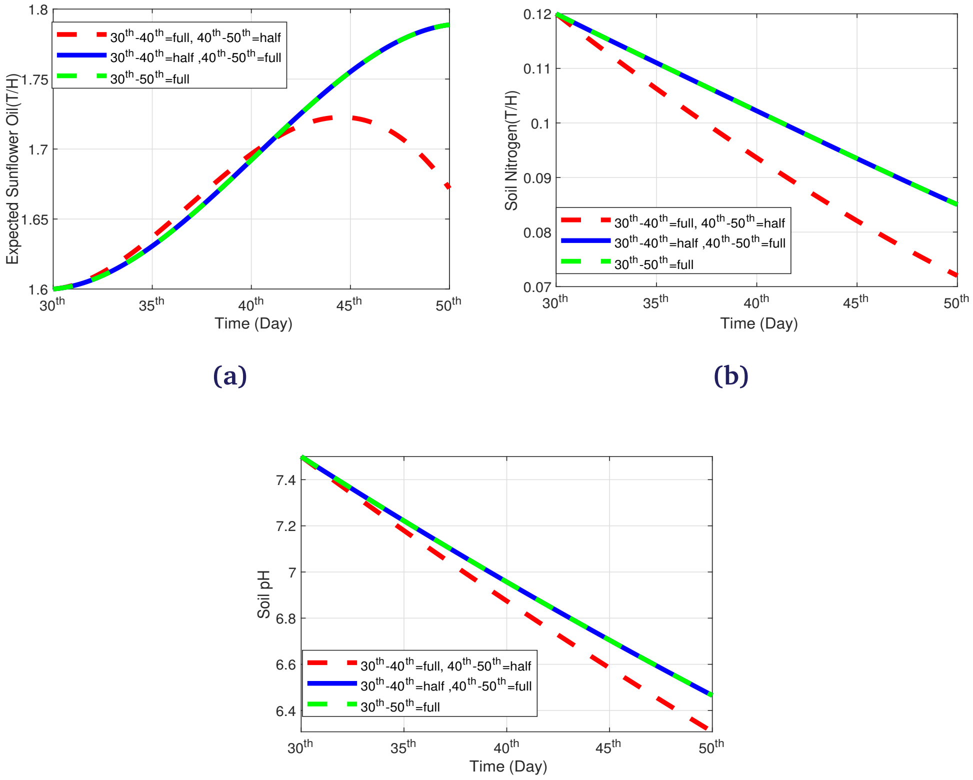
(c)

**Figure 3.** (a) There was an increase in sunflower oil until the nitrogen level remained at the standard value. (b) There was an increase in sunflower oil till the pH level stayed at the standard value; otherwise, there will be a loss in oil. (c) With the soil's pH reaching a standard level, the amount of nitrogen that is present in the soil increases.

because the optimal pH range fosters the activity of beneficial soil bacteria involved in both nitrogen fixation and nitrification. Additionally, a balanced pH facilitates the decomposition of organic matter, releasing organically bound nitrogen back into the soil. Consequently, by maintaining a favorable pH, we not only ensure the optimal functioning of vital microbial processes but also create an environment for maximizing the presence of this essential plant nutrient.

### 4.3. Opinion for Optimistic Production

The predicted yield of sunflower oil after a 30-day growth period is expected to be between 1.6 and 1.8 tonnes per hectare. Figure 4a shows that, when the amount of fertilizer applied is reduced by half after the 1st 10 days, meaning the 40th to 50th day, then it will result in an extreme decrease in oil production. On the other hand, the results from full fertilizer application during all the 20 days (30th to 50th day) and the amount of fertilizer applied is doubled after the 1st 10 days, meaning the 40th to 50th day, (30th to 50th day), the expected oil production is the same. Therefore, applying half fertilizer from the 30th to 40th day and making it double or applying the full amount is more profitable for expected oil production. Figure 4b shows when the amount of fertilizer applied is reduced by half after the 1st 10 days, meaning the 40th to 50th day, then the soil nitrogen level is also decreased. The same way applying half fertilizer from the 30th to 40th day and making it double or applying the full amount is more suitable for nitrogen level of the soil. As shown in Figure 4a, the utilization of fertilizer yields higher profits when applied throughout the period between the 30th and 40th days.



**(a)** **(b)**

**(c)**

**Figure 4.** (a) Predicted sunflower oil differs depending on how fertilizer is used throughout time. (b) The nitrogen content of the soil fluctuates over time in response to fertilizer application techniques. (c) The pH of the soil changes over time depending on how fertilizer is applied.

Figures 4b and 4c demonstrate identical situations once again.

So, Figure 4 shows the importance of timing for fertilizer application in maximizing profits. It suggests that applying fertilizer around the 30th to 40th day after planting leads to a more significant increase in yield and, consequently, greater profit compared to other application times. This suggests that the crop reaches a specific growth stage during this period where it is most receptive to the benefits of fertilizer, resulting in optimal growth and higher yields.

Farmers can use this information to optimize their fertilizer use and maximize profitability, which is of great benefit. Farmers can choose the right sort and amount of fertilizer for their crops. Based on their knowledge of the nutrients that each crop specifically needs as well as the nutrients already present in the soil. This results in reduced waste and improved yields, which increases profitability and saves a substantial amount of money. With this knowledge, farmers can operate more profitably and sustainably, which will eventually help them in the long run. However, Table 2 provides a comparison of related studies and the present work, emphasizing the novel aspects and first-time contributions of this study.

## 5. Conclusion

Nitrogen mismanagement remains a critical problem in sunflower cultivation, as both excess and deficiency reduce yield and oil quality. This study investigates the optimal fertilizer application for maximizing oil production by analyzing plant responses to nitrogen at different vegetative stages through a mathematical model. Numerical analysis reveals that nitrogen must be carefully balanced

**Table 2.** Comparison of related studies and the present work, highlighting its novelty

| Study | Methodology | Focus area | Novelty |
|---|---|---|---|
| Rashidi and Seilsepour [29] | Field experiments and nitrogen application rate analyses | Nitrogen fertilizer efficiency and crop yield | Evaluates optimum nitrogen rates using empirical yield response, demonstrates efficiency effects under different management regimes |
| Zeng et al.[30] | Experimental and statistical modeling | Nitrogen fixation and crop yield relationships | Quantifies effects of nitrogen fixation on crop yield using regression and path analysis |
| Imran et al. [31] | Field experiments and modeling on nitrogen fixation | Nitrogen fixation efficiency in legumes/cereals | Estimates nitrogen fixation efficiency across crop types, compares field and modeling results |
| Present study | System of ODEs modeling nitrogen fixation and yield optimization | Mathematical modeling of nitrogen fixation for sunflower oil production | A newly proposed mathematical model to investigate nitrogen fixation on sunflower oil yield as well as identifying nitrogen management strategies. |

between the 30th and 50th day after planting the sunflower Figure 1. Excessive or insufficient fertilizer results in reduced oil production, especially during this critical phase when the base of the oilseed is formed Figure 4. We used MATLAB (R2018a) to identify equilibrium points and carry out numerical solutions, demonstrating that maintaining nitrogen balance is key to optimizing sunflower oil production.

From a societal perspective, this study provides valuable insights into nitrogen management, which can significantly enhance sunflower cultivation in regions like Bangladesh. With the rise of cardiovascular diseases (CVDs), partly driven by unhealthy diets, sunflower oil offers a healthier alternative due to its high HDL (high-density lipoprotein) content, potentially addressing these health concerns [32, 33]. Moreover, cultivating sunflowers not only improves public health but also fosters economic growth, as the crop thrives in Bangladesh's climate and supports various applications, including food production and skin care. Furthermore, this research will also provide valuable insights to farmers, enabling them to optimize fertilizer use, improve crop yields, and enhance the sustainability of sunflower cultivation. Future work will improve the model by integrating bifurcation analysis, optimal pest dynamics, and microbial soil interactions to better understand the factors that affect sunflower oil production and promote sustainable agriculture in Bangladesh and similar areas.

## Supplementary Information

**Author Contributions.** **Ronjit Mondal:** Conceptualization, investigation, formal analysis, methodology, data collection, software, visualization, Writing–original draft preparation, writing–review and editing. **Jannatul Ferdous Puspo:** Conceptualization, investigation, formal analysis, methodology, data collection, software, visualization, Writing–original draft preparation, writing–review and editing. **Uzzwal Kumar Mallicki:** Conceptualization, formal analysis, methodology, data curation, software, visualization, writing–review and editing, supervision, Validation.

**Acknowledgements.** The authors gratefully acknowledge the editors and reviewers for their insightful comments and thoughtful suggestions, which have significantly contributed to the improvement of this manuscript.

**Funding.** This research received no external funding.

**Conflict of interest.** The authors state that there are no financial interests or personal relationships that could be perceived as influencing the work presented in this study.

**Data availability.** The manuscript contains the data that support the findings of this study.

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