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Critical waves in a nonlocal dispersion delayed susceptible-infected-confined-quarantined-recovered outbreak model with general incidence function

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Abstract. *We study a delayed nonlocal epidemic model that includes the effects of confinement and a generalized incidence function. The model takes into account the spatial movements of the population through a nonlocal dispersal kernel and the behavioral control through a confinement parameter. First, we calculate the basic reproduction number R_0 and show its explicit dependence on the confinement rate. Second, we determine the minimal wave speed ζ^* of traveling wave solutions connecting the disease-free equilibrium to the endemic equilibrium. We show that ζ^* is given by the principal root of the corresponding characteristic equation and that (i) no traveling wave exists for $\zeta < \zeta^*$, and (ii) traveling waves exist for all $\zeta > \zeta^*$. Moreover, we show that the minimal wave speed is a monotone decreasing function of the confinement rate. Our results are obtained through a combination of spectral theory, upper-lower solution methods, and monotone iteration schemes that are modified to account for the joint effects of delay and nonlocal dispersal. Numerical simulations confirm the analytical prediction of the minimal wave speed and illustrate the quantitative slowing effect induced by confinement. These results provide a rigorous mathematical characterization of how mobility, delay, and confinement jointly determine epidemic invasion and spatial propagation.*

1. Introduction

Spatial dynamics of infectious diseases have garnered a lot of attention [1–3], particularly for highly infectious pathogens like the Ebola virus. Traditional compartmental models, such as the SIR model, have been extremely helpful in understanding disease spread. However, these models often cannot capture the complexities brought about by spatial heterogeneity, time delays, and intervention policies like confinement [4–7]. More recent studies have incorporated temporal delays and nonlocal dispersal into epidemic models to solve this deficiency. Nonlocal dispersal more precisely describes the transmission of illness from one spatial point to another by defining a person that travels from one position to another at a different point[8, 9]. However, temporal delays include both the incubation period for the disease and the delay in implementing control measures. As a case in point, Feng and Gao [10] investigated a time-delays nonlocal dispersal SIR model and established the existence of traveling wave solutions and a central role played by the basic reproduction number R_0 on propagation wave speeds.

In terms of disease modeling, a traveling wave solution represents the manner in which the disease is spreading throughout the geographical area in a predictable pattern. If you were to envision a wave moving through a population, you would know that people in front of the wave are healthy and susceptible to disease, people at the front of the wave are newly contracting disease at a rapid rate, and people behind the wave have a well-established and endemic disease. This is analogous to a solution that is moving through space and maintaining its shape at the same time. This speed of the wave, represented by in this work by ζ, indicates the rate at which the epidemic front is moving through space. This is essential information for predicting not only whether a disease will spread through the geographical area, but also the rate at which this spread is taking place [11, 12].

Although some studies, such as [13, 14], have focused on the existence of traveling wave solutions for delayed nonlocal epidemic models, these studies mainly focus on the traditional compartmental models (such as SIR models and SEIR models) without considering the confinement and quarantine strategies at the same time. Moreover, most studies only focus on traditional bilinear incidence rates or specific incidence rates, while the role of a general nonlinear incidence rate with a discrete time delay in determining the minimal wave speed for the existence of traveling wave solutions has not been fully studied.

Moreover, the interplay between nonlocal spatial diffusion, incubation period for the spread of the epidemic, and confinement/quarantine strategies has not been rigorously studied for traveling wave solutions. The existence of additional compartments (C and Q) brings new challenges in the derivation of the characteristic equation for the minimal wave speed for the existence of traveling wave solutions.

In this paper, we shall develop a delayed nonlocal SICQR model for the spread of epidemics with a general incidence rate and derive a precise criterion for the existence/nonexistence of traveling wave solutions. This paper extends the studies in [15, 16] to include confinement-driven models with a general incidence rate within a nonlocal delayed setting.

The addition of confinement policies also complicates epidemic model dynamics. Confinement policies such as lockdowns and quarantines reduce contact rates directly and inhibit the spread of infection [17–20]. It has been shown that these interventions can significantly alter the course of an epidemic, affecting both the rate and shape of disease spread.

In the present paper, we propose a full SICQR model incorporating nonlocal dispersal, time delays, confinement, and effects. Our general aim is to explore the ability of traveling wave solutions within the given model. Of significant interest are the conditions and factors under which the waves would travel and how the parameters influence their behavior. This will be crucial towards determining effective intervention strategies and limiting the spread of infectious diseases. Such mobility can be modeled by considering a nonlocal dispersion operator, which can be defined as follows:

$$\mathfrak{J}[\Phi](x) := J*\Phi(x)-\Phi(x)=\int_{\mathbb{R}} J(x-y)\Phi(y)dy-\Phi(x)=\int_{\mathbb{R}} J(y)\Phi(x-y)dy-\Phi(x), \quad \Phi\in C(\mathbb{R}).$$

In this study, our primary objective is to investigate the existence of traveling wave solutions that connect the disease-free equilibrium (DFE) to the endemic equilibrium (EE) within the framework of a global incidence function and time delay. To this end, we consider the following nonlocal dispersal delayed SICQR epidemic model.

$$\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= d_1(J*S(x,t)-S(x,t))+\Lambda-L(S(x,t),I(x,t))-(\gamma_C-\mu)S(x,t),\\
\frac{\partial I(t,x)}{\partial t} &= d_2(J*I(x,t)-I(x,t))+L(S(x,t-\tau),I(x,t-\tau))-(\delta+\theta+\mu)I((x,t),\\
\frac{\partial C(x,t)}{\partial t} &= d_3(J*C(x,t)-C(x,t))+\theta I(x,t)-(\mu_C+\rho)C(x,t),\\
\frac{\partial Q(x,t)}{\partial t} &= d_4\Delta_x Q(x,t)+\gamma_C S(x,t)-\eta Q(x,t),\\
\frac{\partial R(x,t)}{\partial t} &= d_5(J*R(x,t)-R(x,t))+\delta I(x,t)+\rho C(x,t)+\eta Q(x,t),
\end{aligned} \tag{1}$$

where $t>0$, $x\in\mathbb{R}$, and

$S(x,t)$	:	Density of susceptible individuals at time t and position x.
$I(x,t)$	:	Density of infectious individuals.
$C(x,t)$	:	Density of isolated or hospitalized individuals (under confinement).
$Q(x,t)$	:	Density of quarantined individuals (preventive isolation).
$R(x,t)$	:	Density of recovered individuals.
d_i	:	Diffusion coefficients for each compartment ($i=1,2,3,4,5$).
Λ	:	Entering flux into S-class per unit of time.
μ	:	The natural death coefficient.
τ	:	Time delay representing the incubation period.
θ	:	Rate at which infectious individuals are moved to confinement (C).
δ	:	Recovery rate of infectious individuals.
μ_C	:	Death rate of confined individuals.
ρ	:	Recovery rate from confinement.
γ_C	:	Quarantine rate of susceptible individuals.
η	:	Recovery rate from quarantine.

Then, we make the assumptions:
(A) d_j are positive, and $\mu,\mu_C,\gamma,S,\tau,\rho,\theta,\alpha,\eta,\delta>0$ for $j=1,2,3,4,5$.
(J) $J\in C^1(\mathbb{R})$, $J_i(0)>0$, $J(x)=J(-x)\geq 0$ $\forall x\in\mathbb{R}$, $\int_{\mathbb{R}}J(x)dx=1$; $\lim_{\lambda\to+\infty}\frac{1}{\lambda}\int_{\mathbb{R}}J(y)e^{-\lambda y}dy=+\infty$. Also, we suppose that $L\in C^2(\mathbb{R}^2)$, and
(H) satisfies: $L(S,0)=L(0,I)=0$, $\frac{\partial L(S,I)}{\partial I}>0$, $\frac{\partial^2 L(S,I)}{\partial I^2}<0$ and $\frac{\partial L(S,I)}{\partial S}>0$ for all $S,I>0$.
The first and second equations are independent of C, Q, and R, whereas the latter are dictated by S and I. Therefore, we may remove the C and R equations from the system (1). To understand the dynamics of system (1), we can analyze the system:

$$\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= d_1(J*S(x,t)-S(x,t))+\Lambda-(\mu+\gamma_C)S(x,t))-L(S(x,t),I(x,t)),\\
\frac{\partial I(x,t)}{\partial t} &= d_2(J*I(x,t)-I(x,t))+L(S(x,t-\tau),I(x,t-\tau))-(\mu+\delta+\theta)I(x,t).
\end{aligned} \tag{2}$$

To study the TWS of system (2), we must first find the constant equilibria. $(s^0,0)=(\frac{\Lambda}{\mu+\gamma_C},0)$ is the DFE of system (2), which is always present. To establish positive equilibrium, consider the following ODE system.

$$\begin{aligned}\frac{dS}{dt}&=\Lambda-(\mu+\gamma_C)S(t))-L(S(t),I(t)),\\ \frac{dI}{dt}&=L(S(t),I(t))-(\mu+\delta+\theta)I(t).\end{aligned}\tag{3}$$

The BRN, R_0 given by

$$R_0=\frac{\frac{\partial L(s^0,0)}{\partial I}}{(\mu+\delta+\theta)}.$$

As a result, system (3) enables a unique positive equilibrium. If $s^*,i^*=E^*$, then $R_0>1$ with

$$s^*=\frac{\Lambda-(\mu+\delta+\theta)i^*}{\mu+\gamma_C},$$

i^* is the only positive root of the equation $H(i^*):=L\left(\frac{\Lambda-(\mu+\delta+\theta)i^*}{\mu+\gamma_C},i^*\right)-(\mu+\delta+\theta)i^*=0$. According to **(H)**, $\frac{\partial^2 L(S,I)}{\partial I^2}<0$, which implies that $L(S,I)\le I\frac{\partial L(s^0,0)}{\partial I}$. As a result, $\lim_{I\to 0}H(I)>0$ and $\lim_{I\to+\infty}H(I)=-\infty$, indicating that $H(I)=0$ has at least one positive root. We may investigate the roots of $H'(I)=0$ to determine its uniqueness (see [21]). Furthermore, we get the following result.

Theorem 1. *The DFE is globally asymptotically stable if $R_0<1$, and unstable if $R_0>1$. In the latter case, E^* exists, which is distinctive and globally asymptotically stable.*

2. Existence of Minimal Wave Speed

In the following, we shall always assume $R_0>1$. The system (2) permits two equilibriums: E_0 and E^*. Our primary goal is to confirm the existence of traveling wave solutions of system (1) between the two equilibria E_0 and E^*. The traveling wave solution of system (1) has the following special form:

$$(s(\varepsilon),i(\varepsilon)),\quad \varepsilon=x+\zeta t\in\mathbb{R}.\tag{4}$$

Plugging eq. (4) into system (1) yields the waveform equations as

$$\begin{aligned}\zeta s'(\varepsilon)&=d_1(J*s(\varepsilon)-s(\varepsilon))+\Lambda-(\mu+\gamma_C)s(\varepsilon)-L(s(\varepsilon),i(\varepsilon)),\\ \zeta i'(\varepsilon)&=d_2(J*i(\varepsilon)-i(\varepsilon))+L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)i(\varepsilon),\end{aligned}\tag{5}$$

with the boundary conditions

$$(s,i)(-\infty)=(s^0,0),\quad (s,i)(+\infty)=(s^*,i^*).\tag{6}$$

We aim to find a positive solution of eq. (5) that meets the boundary condition eq. (6). Linearized the second equation of the eq. (5) at $E_0=(s^0,0)$, we get

$$-\zeta i'(\varepsilon)+d_2(J*i(\varepsilon))+\frac{\partial L(s^0,0)}{\partial i}i(\varepsilon-\zeta\tau)-(\mu+\delta+\theta+d_2)i(\varepsilon)=0.\tag{7}$$

By substituting $i(\varepsilon)=e^{\lambda\varepsilon}$ into eq. (7), we derive the following characteristic equation:

$$F(\lambda,\zeta):=-\zeta\lambda+d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda y}dy+\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda}-(\mu+\delta+\theta+d_2)=0.\tag{8}$$

Thus, examining the eq. (8) produces this lemma.

Lemma 2. *We take* $R_0 = \frac{\frac{\partial L(s,i))}{\partial i}|_{(s,i)=(s^0,0)}}{\mu+\delta+\theta} > 1$, *∃* $\zeta^* > 0$ *and* $\lambda^* > 0$ *such that*

$$\frac{\partial F(\lambda,\zeta)}{\partial \lambda^2}\bigg|_{(\lambda^*,\zeta^*)} = 0 \ \ and \ \ F(\lambda^*,\zeta^*) = 0.$$

Furthermore, the following alternatives hold:

(i) *For all* $0 < \zeta < \zeta^*$, *then* $F(\lambda,\zeta) > 0$ *where* $\lambda \in (0,\lambda_\zeta)$, *and* $\lambda_\zeta \in [0,+\infty[$,

(ii) *If* $\zeta > \zeta^*$, *so* $F(\lambda;\zeta) = 0$ *has two roots are positive* $\lambda_1(\zeta) < \lambda_2(\zeta)$ *that satisfy where*

$$\zeta^* = \sup\{\zeta > 0 | F(\lambda,\zeta) > 0, \forall \lambda \in \mathbb{R}\},$$

exists and positive.

Proof. From

$$F(\lambda,\zeta) = -\zeta\lambda + d_2\int_{\mathbb{R}} J(y)e^{-\lambda y}dy + \frac{\partial L(s_0,0)}{\partial i}e^{-\zeta\tau\lambda} - (\mu+\delta+\theta+d_2),$$

we compute

$$\frac{\partial F}{\partial \lambda} = -\zeta - d_2\int_{\mathbb{R}} yJ(y)e^{-\lambda y}dy - \zeta\tau\frac{\partial L(s_0,0)}{\partial i}e^{-\zeta\tau\lambda}.$$

Differentiating once more yields

$$\frac{\partial^2 F}{\partial \lambda^2} = d_2\int_{\mathbb{R}} y^2 J(y)e^{-\lambda y}dy + \zeta^2\tau^2\frac{\partial L(s_0,0)}{\partial i}e^{-\zeta\tau\lambda}.$$

Since $J(y) \geq 0$ and $\frac{\partial L(s_0,0)}{\partial i} > 0$, each term in the second derivative is strictly positive. Hence,

$$\frac{\partial^2 F}{\partial \lambda^2} > 0,$$

which proves that $F(\lambda,\zeta)$ is strictly convex in λ. As $\lambda \to +\infty$, the exponential term $e^{-\zeta\tau\lambda}$ tends to zero, while

$$\int_{\mathbb{R}} J(y)e^{-\lambda y}dy \to +\infty,$$

under assumption (J). Therefore,

$$\lim_{\lambda\to+\infty} F(\lambda,\zeta) = +\infty.$$

At $\lambda = 0$, we have

$$F(0,\zeta) = d_2 + \frac{\partial L(s_0,0)}{\partial i} - (\mu+\delta+\theta) = (\mu+\delta+\theta)(R_0 - 1),$$

which is positive whenever $R_0 > 1$. Then there is $\lambda^* > 0$ satisfying

$$\frac{\partial F(\lambda^*,\zeta)}{\partial \lambda} = -d_2\int_{-\infty}^{+\infty} J(y)ye^{-\lambda^* y}dy - \zeta - \zeta\tau\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda^*} = 0.$$

This implies that λ^* satisfies

$$\frac{-d_2\int_{-\infty}^{+\infty} J(y)ye^{-\lambda^* y}dy - \zeta}{\zeta\tau} = \frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda^*}. \tag{9}$$

Using eq. (9) to simplify $F(\lambda^*,\zeta)=0$, we get

$$F(\lambda^*,\zeta)=-\zeta\lambda^*+d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy+\frac{-d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy-\zeta}{\zeta\tau}-(\mu+\delta+\theta)=0,$$

which equivalent to

$$-\zeta^2\tau\lambda^*+\zeta\left(-\tau(\mu+\rho)+d_2\tau\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy-1\right)-d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy=0. \quad (10)$$

Obtain the corresponding equation

$$F(\lambda,\zeta)=-\zeta\lambda+d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda y}dy+\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda}-(\mu+\delta+\theta)=0. \quad (11)$$

Given the preceding properties of $F(\lambda,\zeta)$, obtain the following findings for $\zeta>0$.

$$\begin{aligned}
\frac{\partial F(\lambda,\zeta)}{\partial\lambda} &= -d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy-\zeta-\zeta\tau\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda},\\
\frac{\partial^2 F(\lambda,\zeta)}{\partial\lambda^2} &= d_2\int_{-\infty}^{+\infty}J(y)y^2e^{-\lambda y}dy+\zeta^2\tau^2\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda}>0,\\
\lim_{l\to+\infty}F(\lambda,\zeta) &= +\infty,\\
F(0,\zeta) &= d_2+\frac{\partial L(s^0,0)}{\partial i}-\mu-\delta-\theta\\
&= (\mu+\delta+\theta)[R_0-1]>0.
\end{aligned}$$

Then there is $\lambda^*>0$ such that

$$\frac{\partial F(\lambda,\zeta)}{\partial\lambda}|_{\lambda=\lambda^*}=-d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy-\zeta-\zeta\tau\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda^*}=0,$$

hence

$$\frac{-d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy-\zeta}{\zeta\tau}=\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda^*},$$

when we substitute the result in $F(\lambda^*,\zeta)$, we obtain

$$F(\lambda^*)=-\zeta\lambda^*+d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy+\frac{-d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy-\zeta}{\zeta\tau}-(\mu+\delta+\theta)=0,$$

hence,

$$-\zeta^2\tau\lambda^*+\zeta\left(-\tau(\mu+\rho+\theta_2(1-\varepsilon))+d_2\tau\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy-1\right)-d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy=0.$$

Then

$$\zeta^*=\frac{1}{2\tau\lambda^*}\left[-\left(-\tau(\mu+\delta+\theta)+d_2\tau\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy-1\right)\right.$$

$$+\sqrt{\left(-\tau(\mu+\delta+\theta)+d_2\tau\int_{-\infty}^{+\infty}J(y)e^{-\lambda^* y}dy+1\right)^2+4\tau\lambda^*(d_2\int_{-\infty}^{+\infty}J(y)ye^{-\lambda^* y}dy)}\Bigg]>0.$$

Then we present (i)-(ii). A simple calculation yields

$$\begin{aligned}
F(\lambda,0)&=d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda y}dy+\frac{\partial L(s^0,0)}{\partial i}-(\mu+\delta+\theta)\\
&=d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda y}dy+(\mu+\delta+\theta)[R_0-1]>0,\\
\frac{\partial F(\lambda,\zeta)}{\partial\zeta}&=-\lambda-\tau\lambda\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda}<0,\ \ \forall\lambda>0,\\
\lim_{\zeta\to+\infty}F(\lambda,\zeta)&=-\infty.
\end{aligned}$$

For (i), if $0<\zeta<\zeta^*$, we have $F(\lambda,\zeta)>0$. If $\zeta>\zeta^*$, $F(\lambda,\zeta)=0$ has two positive real roots. $\lambda_1(\zeta)<\lambda_2(c)$ with

$$F(\lambda,\zeta)>0\quad\text{where}\quad\lambda\in(0,\lambda_1(\zeta))\cup(\lambda_2(\zeta),\infty),$$

and

$$F(\lambda,\zeta)<0\quad\text{where}\quad\lambda\in(\lambda_1(\zeta).\lambda_2(\zeta)).$$

For (ii) If $\zeta>0$, hence

$$\begin{aligned}
F(0,\zeta)&=d_2+\frac{\partial L(s^0,0)}{\partial i}-(\mu+\delta+\theta)>0,\\
\frac{\partial F(0,\zeta)}{\partial\lambda}&=-d_2\int_{-\infty}^{+\infty}J(y)ydy-\zeta-\zeta\frac{\partial L(s^0,0)}{\partial i}\tau<0,
\end{aligned}$$

we note that

$$\frac{\partial^2F(\lambda,\zeta)}{\partial\lambda^2}=d_2\int_{-\infty}^{+\infty}J(y)y^2e^{-\lambda y}dy+\zeta^2\tau^2\frac{\partial L(s^0,0)}{\partial i}e^{-\zeta\tau\lambda}>0.$$

This implies that F is convex in λ, and the existent of $\lambda^*>0$ indicates λ_1,λ_2. ■

To explain the lack of a traveling wave solution, consider the following sections.

3. Criteria for the Nonexistence of Traveling Waves

The theorem below illustrates a case where the eq. (5) does not accept a traveling wave solution.

Theorem 3. *If $R_0>1$ and $0<\zeta<\zeta^*$, hence, crefsir3 has no TWS of the form $(s(\varepsilon),i(\varepsilon))$ that satisfies eq.* (6).

Proof. Assume a traveling wave $(s^*(\varepsilon),i^*(\varepsilon))$ of eq. (5) that meets criteria eq. (6) for some $0<\zeta<\zeta^*$. According to eq. (6) and $R_0>1$, for each given $\lambda>0$, there exists some $M_\lambda>0$ big enough so that $s^0-\lambda\le s^*(\varepsilon)<s^0$ for all $\varepsilon\le -M_\lambda$. Using the equation $i(\varepsilon)$ of the eq. (5), then we have

$$\begin{aligned}
\zeta i^*(\varepsilon)&=d_2(J*i^*(\varepsilon)-i^*(\varepsilon))+L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)i^*(\varepsilon),\\
&\ge d_2(J*i^*(\varepsilon)-i^*(\varepsilon))+L(s^0-\lambda,i(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)i^*(\varepsilon).
\end{aligned}\tag{12}$$

Consider $\varepsilon < -M_\lambda$. Given the conditions (6), for all δ and M^0 are positive such that $s^*(\varepsilon) \geq \delta$ and $i^*(\varepsilon) \leq M^0$, $\forall \varepsilon \in \mathbb{R}$. Using **(H)**, we see that

$$\frac{L(s^0-\lambda, i^*(\varepsilon-\zeta\tau))i^*(\varepsilon-\zeta\tau)}{L(\delta, i^*(\varepsilon-\zeta\tau))i^*(\varepsilon-\zeta\tau)} = \frac{L(s^0-\lambda, i^*(\varepsilon-\zeta\tau))}{L(\delta, i^*(\varepsilon-\zeta\tau))} \geq \frac{L((s^0-\lambda, i^*(\varepsilon-\zeta\tau)))}{L(s*(\varepsilon-\zeta\tau), i^*(\varepsilon-\zeta\tau))},$$
$$\leq \frac{M^0}{L(\delta, M^0)}\frac{\partial L(s^0,0)}{\partial i} < \infty, \quad \varepsilon > -M_\lambda.$$

Given that $i^*(\varepsilon) > 0$ for $\varepsilon \in \mathbb{R}$ and $i^*(+\infty) = i^* > 0$, there exists a positive constant $i^-(\varepsilon) > 0$ such that $i^*(\varepsilon) \geq i^-$ for all $\varepsilon \geq -M_\lambda$. Thus, we may select a constant $h > 1$ so that

$$\frac{L(s^0-\lambda, i^*(\varepsilon-\zeta\tau))}{(1+i^*(\varepsilon-\zeta\tau))^h} \leq L(s^*(\varepsilon-\zeta\tau), i^*(\varepsilon-\zeta\tau)) \quad \text{for} \quad \varepsilon > -M_\lambda.$$

For $\varepsilon > -M_\lambda$, the following inequality applies:

$$\zeta i^*(\varepsilon) \geq d_2(J * i^*(\varepsilon) - i^*(\varepsilon)) + \frac{L(s^*(\varepsilon-\zeta\tau), i^*(\varepsilon-\zeta\tau))}{(1+i(\varepsilon-\zeta\tau))^h} - (\mu+\delta+\theta)i^*(\varepsilon). \tag{13}$$

Combining eq. (12) and eq. (13), we finally obtain

$$\zeta i^*(\varepsilon) \geq d_2(J * i^*(\varepsilon) - i^*(\varepsilon)) + \frac{L(s^0-\lambda, i^*(\varepsilon-\zeta\tau))}{(1+i^*(\varepsilon-\zeta\tau))^h} - (\mu+\delta+\theta)i^*(\varepsilon), \quad \varepsilon \in \mathbb{R}. \tag{14}$$

Let $b(u) = \inf_{u \leq v \leq M^0} \frac{L(s^0-\varepsilon, v)}{(1+v)^h}$, $u(x,t) = i^*(x+\zeta\tau)$. According to eq. (14),

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t} \geq d_2(J * u(x,t) - u(x,t)) + b(u(x,t-\tau)) - (\delta+\mu+\theta)u(x,t-\tau),\\ &u(x,t) = i^*(x+\zeta\tau), \quad x \in \mathbb{R}, t > 0.\end{aligned}$$

Using the comparison principle [22], we get

$$u(x,t) \geq v(x,t), \quad x \in \mathbb{R}, t \geq 0, \tag{15}$$

with $v(x,t)$ check the equation:

$$\begin{aligned}&\frac{\partial v(x,t)}{\partial t} = d_2(J * v(x,t) - v(x,t)) + b(v(x,t-\tau)) - (\delta+\mu+\theta)v(x,t-\tau),\\ &v(x,t) = i^*(x+\zeta\tau), \quad x \in \mathbb{R}, t > 0.\end{aligned} \tag{16}$$

For any $\hat{\zeta} \in (0, \zeta^*)$

$$\lim_{t\to\infty} \inf_{|x| \leq \hat{\zeta} t} v(x,t) > 0. \tag{17}$$

Using the asymptotic spreading theory [23], we analyze the dynamics of eq. (16). The operator $J * \cdot$ generates a C_0-semigroup, as established in [24, 25]. The eq. (16) is an ordinary differential equation (ODE) with exactly two equilibria: the trivial equilibrium 0, and $v = v^* > 0$, where v^* satisfies the algebraic condition

$$b(v^*) - (\mu+\delta+\theta)v^* = 0.$$

We define the phase space $C := C(\mathbb{R} \times [-\tau, 0])$, and let

$$C_{v^*} := \{v \in C : 0 \leq v \leq v^*\}.$$

According to semigroup theory [24, 25], eq. (16) generates a monotone semiflow $Q^t : C_{v^*} \to C_{v^*}$, which is defined as follows. For $\zeta > -M_\varepsilon$, the following inequality holds:

$$Q^t(\psi)(x) = v(x+t,\tau), \quad x \in \mathbb{R}, t, \tau \geq 0, \psi \in C_{v_i^*}.$$

Let $v(x,t)$ be the unique solution of eq. (16) with initial condition $v(x,t-\tau) = \psi$. Define the function space $\tilde{C} := C([-\tau, 0])$, and let

$$\tilde{C}_{v^*} := \{v \in \tilde{C} : 0 \leq v \leq v^*\}.$$

We denote by $\tilde{Q}^t : \tilde{C}_{v^*} \to \tilde{C}_{v^*}$ the solution semiflow associated with the following delayed differential equation:

$$\frac{dv(t)}{dt} = b(v(t-\tau)) - (\delta + \mu + \theta)v(t-\tau), \quad t > 0.$$

Using the starting value $v^0 = \psi^0 \in \tilde{C}_{v^*}$, where $v^t = v(t-\tau)$. According to [26]'s Corollary 5.3.5, $\tilde{Q}_t$ becomes highly monotone on $\tilde{C}_{v^*}$. Using the Dancer-Hess connecting orbit lemma [27], we can conclude that $\tilde{Q}_t$ is a strongly monotone full orbit that connects 0 to v_i^*. Hence, hypothesis (A5) in [22] holds. In fact, we can readily demonstrate that for any $t > 0$, $\tilde{Q}_t$ meets all hypotheses in [22] (A1-A5). It is apparent that $\tilde{Q}_t$ fulfils eq. (16). Therefore, $\tilde{Q}_t$ is the limitation of Q_t to $\tilde{C}_{v^*}$. This indicates that Theorem 2.17 from [22] may be used. Thus, eq. (17) holds.
Taking $\zeta_0 \in (\zeta, \zeta^*)$ and we let $x = -\zeta_0 t$, it follows from eqs. (15) and (17) that

$$\liminf_{t\to\infty} u(x,t) \geq \liminf_{t\to\infty, |x|\leq \zeta_0 t} v(x,t) > 0. \tag{18}$$

Since $\varepsilon = x + \zeta t = (\zeta - \zeta_0)t \to -\infty$ as $t \to \infty$, we finally obtain

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} i^*(x+\zeta t) = \lim_{t\to\infty} i^*((\zeta - \zeta_0)t) = \lim_{\varepsilon\to-\infty} i^*(\varepsilon) = 0,$$

contradiction with eq. (18). This completes the proof. ■

Remark 1. The critical value ζ^* represents the minimal spatial propagation speed of the epidemic front. Biologically, ζ^* quantifies how fast the infection spreads geographically when introduced into a susceptible population. The existence of a minimal speed indicates that:

- For $\zeta < \zeta^*$, spatial invasion fails and no coherent epidemic front can form;
- For $\zeta > \zeta^*$, the disease propagates as a traveling infection wave.

The dependence of ζ^* on model parameters provides epidemiological insight:

- Increasing the confinement rate θ reduces ζ^*, slowing the geographical expansion of the epidemic;
- Larger delay τ may either accelerate or decelerate propagation depending on its interaction with transmission intensity;
- Broader dispersal kernels J (long-distance mobility) increase ζ^*, reflecting faster spatial spread.

Therefore, ζ^* serves as a quantitative indicator of epidemic expansion speed under spatial mobility and control interventions.

4. Existence of Traveling Wave

4.1. Sub-super solution

If $\zeta > \zeta^*$, we create a upper-lower solutions of eq. (5) using an iterative approach.

Definition 4. Denoted (s^+, i^+) and (s^-, i^-) the pair of supper and sub solutions of eq. (5), respectively, and satisfy

$$-\zeta(s^+)'(\varepsilon) + d_1(J * s(\varepsilon) - s(\varepsilon)) + \Lambda - (\mu + \gamma_C)(s^+)(\varepsilon) - L((s^+)(\varepsilon), (i^-)(\varepsilon)) \leq 0, \tag{19}$$

$$-\zeta(s^-)'(\varepsilon)+d_1(J*s(\varepsilon)-s(\varepsilon))+\Lambda-(\mu+\gamma_C)(s^-)(\varepsilon)-L((s^-)(\varepsilon),(i^+)(\varepsilon))\geq 0, \tag{20}$$

$$-\zeta(i^+)'(\varepsilon)+d_2(J*i(\varepsilon)-i(\varepsilon))+L((s^+(\varepsilon-\zeta\tau)),(i^+)(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)(i^+)(\varepsilon)\leq 0, \tag{21}$$

$$-\zeta(i^-)'(\varepsilon)+d_2(J*i(\varepsilon)-i(\varepsilon))+L((s^-(\varepsilon-_c\tau)),(i^-)(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)(i^-)(\varepsilon)\geq 0, \tag{22}$$

except for finite points of $\varepsilon\in\mathbb{R}$.

Lemma 5. *We take $R_0>1$, and $\zeta>\zeta^*$. Let*

$$s^+(\varepsilon)=s^0, \qquad i^+(\varepsilon)=\min\{e^{\lambda_1\varepsilon},\mathscr{B}\},$$
$$s^-(\varepsilon)=\max\left\{s^0-Me^{\kappa\varepsilon},0\right\}, \quad i^-(\varepsilon)=\max\{e^{\lambda_1\varepsilon}(1-Le^{\eta\varepsilon}),0\},$$

with $\kappa,L>0$, eqs. (19) to (22) are fulfilled. $\mathscr{B}>0$ is the smallest root of $L(s^0,\vartheta)=(\mu+\delta+\theta)\vartheta$, on $\mathbb{R}^+$.

Proof.

(i) Clearly $s^+(\varepsilon)=s^0$ satisfies

$$-\zeta(s^+)'(\varepsilon)+d_1(J*s^+(\varepsilon)-s^+(\varepsilon))+\Lambda-(\mu+\gamma_C)s^+(\varepsilon)-L(s^+(\varepsilon),i^-(\varepsilon)) \quad \leq \quad 0, \tag{23}$$

then, eq. (19) is satisfied.

(ii) For $\varepsilon<\varepsilon_0$, where $\varepsilon_0=\frac{\ln\mathscr{B}}{\lambda_1}$, we have $i^+(\varepsilon)=\mathscr{B}$, and then $i^+(\varepsilon-\zeta\tau)\leq\mathscr{B}$. So, we have

$$\begin{aligned}0&=L(s^0,\mathscr{B})-(\mu+\delta+\theta)\mathscr{B},\\ &\geq d_2(J*i(\varepsilon)-i(\varepsilon))+L(s^+(\varepsilon-\zeta\tau),i^+(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)i^+(\varepsilon)-\zeta(i^+)'(\varepsilon).\end{aligned}$$

For $\varepsilon>\varepsilon_0$, we obtain $i^+(\varepsilon)=e^{\lambda_1\varepsilon}$, which we demonstrate fulfils eq. (21). It is easy to check that.

$$\begin{aligned}\mathscr{M}&\leq d_2(J*i^+(\varepsilon)-i^+(\varepsilon))+\frac{\partial L(s^0,0)}{\partial i}(i^+)(\varepsilon-\zeta\tau)-(\mu+\delta+\theta)(i^+)(\varepsilon)-\zeta(i^+)'(\varepsilon),\\ &\leq -\zeta(i^+)'(\varepsilon)+d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda_1 y}dy+\frac{\partial L(s^0,0)}{\partial i}i^+(\varepsilon-\zeta\tau)-(\mu+\delta+d_2+\theta)i^+(\varepsilon),\\ &= d_2\int_{-\infty}^{+\infty}J(y)e^{-\lambda_1 y}dy+\frac{\partial L(s^0,0)}{\partial i}e^{\lambda_1(\varepsilon-\zeta\tau)}-(\mu+\delta+\theta+d_2)e^{\lambda_1\varepsilon}-\zeta\lambda_1e^{\lambda_1\varepsilon},\\ &= e^{\lambda_1\varepsilon}F(\lambda_1,\zeta),\\ &=0,\\ \mathscr{M}&=d_2(J*i^+(\varepsilon)-i^+(\varepsilon))+L((s^+)(\varepsilon-\zeta\tau),i^+(\varepsilon-\zeta\tau))-(\mu+\delta+\theta)(i^+)(\varepsilon)-\zeta(i^+)'(\varepsilon),\end{aligned} \tag{24}$$

use the definition of λ_1.

(iii) Choosing $0<\gamma<\min\left\{\lambda_1,\frac{\zeta}{d_2}\right\}$. Suppose that $\varepsilon\neq\frac{1}{\gamma}\ln\frac{1}{M}:=\varepsilon^*$, and we claim that s^- satisfies

$$-\zeta(s^-)'(\varepsilon)+d_1(J*s^-(\varepsilon)-s^-(\varepsilon))+\Lambda-(\mu+\gamma_C)(s^-)(\varepsilon)-Le^*(s^-(\varepsilon),i^+(\varepsilon))\geq 0.$$

To establish this assertion, we first assume that $\varepsilon>\varepsilon^*$. This means that $s^-(\varepsilon)=0$ in (ε^*,∞), therefore the inequality holds immediately. If $\varepsilon<\varepsilon^*$, we get $s^-(\varepsilon)=s^0-Me^{\gamma\varepsilon}$. The concavity of $L(s(\varepsilon),i(\varepsilon)$ yields $L(s(\varepsilon),i(\varepsilon)\leq\frac{\partial L(s^0,0)}{\partial i}i(\varepsilon)$. Finally, we have

$$\begin{aligned}\mathscr{N}&\geq 0,\\ &\geq \zeta M\gamma e^{\gamma\varepsilon}+d_1Me^{\gamma\varepsilon}\int_{-\infty}^{+\infty}J(x)e^{-\gamma x}dx+\Lambda-(\mu+\gamma_C)(s^0-Me^{\gamma\varepsilon})-\frac{\partial L(s^0,0)}{\partial i}e^{\lambda\varepsilon}),\\ &= e^{\gamma\varepsilon}\left[\zeta M\gamma-d_1M\int_{-\infty}^{+\infty}J(x)e^{-\gamma x}dx+d_1M-\frac{\partial L(s^0,0)}{\partial i}\left(\frac{s^0}{M}\right)^{\frac{\lambda-\gamma}{\gamma}}\right],\\ \mathscr{N}&= -\zeta(s^-)'(\varepsilon)+d_1(J*s^-(\varepsilon)-s^-(\varepsilon))+\Lambda-(\mu+\gamma_C)s^-\varepsilon)-L(s^-(\varepsilon),i^+(\varepsilon)).\end{aligned}$$

Here we use

$$e^{\lambda\varepsilon} < \left(\frac{s^0}{M}\right)^{\frac{\lambda-\gamma}{\gamma}} \quad \text{for} \quad \varepsilon < \varepsilon^*.$$

Keeping $\gamma M = 1$, letting $M \to \infty$ for some $M > s^0$ large enough and γ small enough, we have

$$-\zeta(s^-)'(\varepsilon) + d_1(J * s^-(\varepsilon) - s^-(\varepsilon)) + \Lambda - (\mu + \gamma_C)(s^-)(\varepsilon) - L(s^-(\varepsilon), i^+(\varepsilon)) \geq 0.$$

The claim is proved.

(iv) Selecting $0 < \eta < \min\{\lambda_2 - \lambda_1, \lambda_1\}$ and $L > 0$ suitably big. Next, we assert that $i^-(\varepsilon)$ satisfies

$$-\zeta(i^-)'(\varepsilon) + d_2(J * s(\varepsilon) - s(\varepsilon)) + L(s^-(\varepsilon - \zeta\tau), i^-(\varepsilon - \zeta\tau)) - (\mu + \delta + \gamma_C)i^-(\varepsilon) \geq 0, \tag{25}$$

with $\varepsilon \neq \varepsilon_2 := \frac{-\ln L}{\eta}$.

We demonstrate this claim in two independent cases: $\varepsilon > \varepsilon_2$ and $\varepsilon < \varepsilon_2$. If $\varepsilon > \varepsilon_2$, then $i^-(\varepsilon) = 0$, indicating that (25) is met. If $\varepsilon < \varepsilon_2$, we have $i^-(\varepsilon) = e^{\lambda_1\varepsilon}(1 - Le^{\eta\varepsilon})$. In this situation, we demonstrate that (25) holds for sufficiently big L, which will be established later. The inequality in (25) may be represented as follows.

$$\begin{aligned}\frac{\partial L(s^0,0)}{\partial i} i^-(\varepsilon - \zeta\tau) - L(s^-(\varepsilon - \zeta\tau), i^-(\varepsilon - \zeta\tau)) \leq & -\zeta(i^-)'(\varepsilon) + d_2(J * i^-(\varepsilon) - i^-(\varepsilon)) \\ & + \frac{\partial L(\frac{\Lambda}{\mu},0)}{\partial i} i^-(\varepsilon - \zeta\tau) - (\mu + \delta + \theta)i^-(\varepsilon), \\ \leq & -LF(\lambda_1 + \eta, \zeta)e^{(\lambda_1+\eta)\varepsilon}.\end{aligned} \tag{26}$$

For any $\xi \in (0, \frac{\partial L(s,i)}{\partial i}|_{(s,i)=(s^0,0)})$, $L(s,i)/i$ is a decreasing function on $(0,\infty)$. Given that i^- is a bounded function for $\varepsilon < \varepsilon_2$, there exists $\delta_0 > 0$ such that $0 < i^- < \delta_0$ for every $\varepsilon < \varepsilon_2$. The boundedness of i^- for $\varepsilon < \varepsilon_2$, and the fact that $\frac{\partial L(s,i)}{\partial i}|_{(s,i)=(s^0,0)} > 0$ implies the existence of $\xi > 0$ small as necessary in such a way the following inequality. For every $0 < i^- < \delta_0$, $L(s^-, i^-)/i^- \geq \frac{\partial L(s^0,0)}{\partial i} - \xi > 0$. Using the knowledge that $0 < i^- < \delta_0$, we get

$$\begin{aligned}\frac{\partial L(s^0,0)}{\partial i} i^-(\varepsilon - \zeta\tau) - L(s^-(\varepsilon - \zeta\tau), i^-(\varepsilon - \zeta\tau)) &= \left(\frac{\partial L(\frac{\Lambda}{\mu+\gamma_C},0)}{\partial i} - \frac{L(s^-(\varepsilon - \zeta\tau), i^-(\varepsilon - \zeta\tau))}{i^-(\varepsilon - \zeta\tau)}\right) i^-(\varepsilon - \zeta\tau), \\ &\leq \left(\frac{\frac{\partial L(\frac{\Lambda}{\mu+\gamma_C},0)}{\partial i} - \frac{L(s^-(\varepsilon-\zeta\tau), i^-(\varepsilon-\zeta\tau))}{i^-(\varepsilon-\zeta\tau)} + i^-(\varepsilon - \zeta\tau)}{2}\right)^2, \\ &\leq \left[\frac{\partial L(s^0,0)}{\partial i} - \left(\frac{\partial L(s^0,0)}{\partial i} - \xi\right) + i^-(\varepsilon - \zeta\tau)\right]^2.\end{aligned} \tag{27}$$

Then, we have

$$\frac{\partial L(s^0,0)}{\partial i} i^-(\varepsilon - \zeta\tau) - L(s^-(\varepsilon - \zeta\tau), i^-(\varepsilon - \zeta\tau)) \leq i^{-2}(\varepsilon - \zeta\tau).$$

Therefore, to prove the inequality (26), it is sufficient to show that

$$(i^{-2})(\varepsilon - \zeta\tau) \leq -LF(\lambda_1 + \eta, \zeta)e^{(\lambda_1+\eta)\varepsilon}. \tag{28}$$

Noting that $i^- \leq i^+$, then we have $(i^-(\varepsilon - \zeta\tau))^2 \leq e^{2\lambda_1\varepsilon}$. To ensure eq. (28), we show that

$$e^{2\lambda_1\varepsilon} \leq -LF(\lambda_1 + \eta, \zeta)e^{(\lambda_1+\eta)\varepsilon}. \tag{29}$$

Because both sides of the inequality (29) are limited for all $\varepsilon < \varepsilon_2$ and trend to 0 as $\varepsilon \to -\infty$, the inequality (29) is valid for sufficiently big M. The proof is complete.

■

4.2. Truncated problem

We put $\nabla > \max\{|\varepsilon^*|, |\varepsilon_0|, r\}$, we take the following bounded set

$$\Gamma_\nabla(\varepsilon) = \Big\{ (\phi(\varepsilon)\varphi(\varepsilon)) \in C\left([-\nabla,\nabla],\mathbb{R}^2\right) \Big| \phi(-\nabla) = s(-\nabla), \\ \varphi(-\nabla) = i(-\nabla),\ \ s^-(\varepsilon) \le \phi(\varepsilon) \le s^0,\ \ i^-(\varepsilon) \le \varphi(\varepsilon) \le i^+(\varepsilon), \\ \varepsilon \in [-\nabla,\nabla] \Big\}.$$

For any $(\phi,\varphi(\varepsilon)) \in \Gamma_\nabla(\varepsilon)$, we define

$$\hat{\phi}(\varepsilon) = \begin{cases} \phi(\nabla), & \varepsilon > \nabla, \\ \phi(\varepsilon), & |\varepsilon| \le \nabla, \\ s^-(-\nabla), & \varepsilon < -\nabla, \end{cases} \qquad \hat{\varphi}(\varepsilon) = \begin{cases} \varphi(\nabla), & \varepsilon > \nabla, \\ \varphi(\varepsilon), & |\varepsilon| \le \nabla, \\ i^-(-\nabla), & \varepsilon < -\nabla, \end{cases}.$$

Clearly, $\Gamma_\nabla(\varepsilon)$ is a closed and convex set. The expression $\left(\hat{\phi}(\varepsilon), \hat{\varphi}(\varepsilon)\right)$ is

$$s^-(\varepsilon) \le \hat{\phi}(\varepsilon) \le s^0, \quad i^-(\varepsilon) \le \hat{\varphi}(\varepsilon) \le i^+(\varepsilon), \quad \varepsilon \in \mathbb{R}.$$

Letting the truncated problem

$$\begin{aligned} \zeta s'(\varepsilon) &= d_1((J*\hat{\phi})(\varepsilon) - s(\varepsilon)) + \Lambda - (\mu+\gamma_C)s(\varepsilon) - L(s(\varepsilon),\varphi(\varepsilon)), \\ \zeta i'(\varepsilon) &= d_2((J*\hat{\varphi})(\varepsilon) - i(\varepsilon)) + L(\hat{\phi}(\varepsilon-\zeta\tau),\hat{\varphi}(\varepsilon-\zeta\tau)) - ((\mu+\theta)+\delta+)i(\varepsilon), \end{aligned} \tag{30}$$

with

$$s(-\nabla) = s^-(-\nabla), \quad i(-\nabla) = i^-(-\nabla). \tag{31}$$

The general results of differential equations ensure that the starting value problems eqs. (30) and (31) permit a single nonnegative solution $(s_\nabla(\varepsilon), i_\nabla(\varepsilon))$ defined for $\varepsilon \in [-\nabla,\nabla]$. Therefore, we let the solution map $\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2)$ on $\Gamma_\nabla(\varepsilon)$ by

$$\mathscr{X}_1(\phi,\varphi) = s_\nabla, \qquad \mathscr{X}_2(\phi,\varphi) = i_\nabla.$$

Lemma 6. *For all $\nabla > \max\{|\varepsilon^*|, |\varepsilon_0|, r\}$, map $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) : \Gamma_\nabla(\varepsilon) \to \Gamma_\nabla(\varepsilon)$.*

Lemma 6 may be inferred from Lemma 5 and the comparison principle. See [12].

Lemma 7. *The map $\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2) : \Gamma_\nabla(\varepsilon) \to \Gamma_\nabla(\varepsilon)$ is completely continuous.*

Proof. For any $(\phi,\varphi) \in \Gamma_\nabla(\varepsilon)$, we readily establish from eq. (30) that $(s_\nabla(\varepsilon), i_\nabla(\varepsilon)) \in C^1([-\nabla,\nabla],\mathbb{R}^2)$. The Arzelà-Ascoli theorem implies that map $\mathscr{X}$ is compact. Next, we look at the continuity of $\mathscr{X}$. Define $s_{\nabla,k}(\varepsilon) = \mathscr{F}_1(\phi_k,\varphi_k)(\varepsilon).$, $i_{\nabla,k}(\varepsilon) = \mathscr{X}_2(\phi_k,\varphi_k)(\varepsilon)$ where $(\phi_k(\varepsilon),\varphi_k(\varepsilon)) \in \Gamma_\nabla(\varepsilon)$ $(k=1,2)$ for $\varepsilon \in [-\nabla,\nabla]$. We first examine the continuity of $\mathscr{X}_1$. The first equation of eq. (30) produces

$$\begin{aligned} \zeta(s'_{\nabla,1}(\varepsilon) - s'_{\nabla,2}(\varepsilon)) + (d_1+\mu+\gamma_C)(s_{\nabla,1}(\varepsilon) - s_{\nabla,2}(\varepsilon)) &= d_1\int_{\mathbb{R}} J(y)(\hat{\phi}_1(\varepsilon-y) - \hat{\phi}_2(\varepsilon-y))dy \\ &\quad + L(s_{\nabla,2}(\varepsilon),\varphi_2(\varepsilon)) - L(s_{\nabla,1}(\varepsilon),\varphi_1(\varepsilon)). \end{aligned} \tag{32}$$

Since

$$\int_{\mathbb{R}} J(y)\hat{\phi}(\varepsilon-y)dy = \int_{-\infty}^{-\nabla} J(\varepsilon-y)s(y)dy + \int_{-\nabla}^{\nabla} J(\varepsilon-y)\phi(y)dy + \int_{\nabla}^{+\infty} J(\varepsilon-y)\phi(\nabla)dy,$$

we have

$$\left|\int_{\mathbb{R}} J(y)(\hat{\phi}_1(\varepsilon-y) - \hat{\phi}_2(\varepsilon-y))dy\right| \le 2\max_{y\in[-\nabla,\nabla]} |\phi_1(y) - \phi_2(y)|. \tag{33}$$

For $(\phi_1,\varphi_1),(\phi_2,\varphi_2)\in\Gamma_\nabla(\varepsilon)$, since $i^+(\varepsilon)\le\mathscr{B}$ for $\varepsilon\in[-X,X]$, then

$$\left|L(\phi_1(\varepsilon),\varphi_1(\varepsilon)-L(\phi_2(\varepsilon),\varphi_2(\varepsilon))\right|\le M_4\left[\left|\phi_1(\varepsilon)-\phi_2(\varepsilon)\right|+\left|\varphi_1(\varepsilon)-\varphi_2(\varepsilon)\right|\right],\tag{34}$$

where $M_4=\sup\left\{\frac{\partial L(s^0,0)}{\partial i},L(\sigma,\mathscr{B}):0\le\sigma\le s^0\right\}$.
Let $u(\varepsilon)=\zeta|s_{\nabla,1}(\varepsilon)-s_{\nabla,2}(\varepsilon)|$. Then, from eqs. (32) to (34), we obtain

$$\begin{aligned}u'(\varepsilon)&=\zeta\,\mathrm{sign}(s_{\nabla,1}(\varepsilon)-s_{\nabla,2}(\varepsilon))(s'_{\nabla,1}(\varepsilon)-s'_{\nabla,2}(\varepsilon)),\\&\le 2d_1\max_{y\in[-\nabla,\nabla]}|\phi_1(y)-\phi_2(y)|-(d_1+(\mu+\gamma_C)-M_4)|s_{\nabla,1}(\varepsilon)-s_{\nabla,2}(\varepsilon)|\\&\quad+M_4|\varphi_2(\varepsilon)-\varphi_1(\varepsilon)|,\\&=\left(\frac{d_1+(\mu+\gamma_C)}{\zeta}+\frac{M_4}{\zeta}\right)u(z)+2d_1\max_{y\in[-\nabla,\nabla]}|\phi_1(y)-\phi_2(y)|+M_4|\varphi_2(\varepsilon)-\varphi_1(\varepsilon)|.\end{aligned}$$

Thus, $\forall\varepsilon\in[-\nabla,\nabla]$, we obtain

$$\begin{aligned}u(\varepsilon)\le u(-\nabla)e^{-\left(\frac{d_1+\mu+\gamma_C}{\zeta}+\frac{M_4}{\zeta}\right)(\varepsilon+\nabla)}+\int_{-\nabla}^{\varepsilon}\Bigg[\Bigg(2d_1\max_{y\in[-\nabla,\nabla]}|\phi_1(y)-\phi_2(y)|\Bigg)\\+M_4\max_{y\in[-\nabla,\nabla]}|\varphi_1(y)-\varphi_2(y)|\Bigg]e^{-\left(\frac{d_1+(\mu+\gamma_C)}{\zeta}+\frac{M_4}{\zeta}\right)(\varepsilon-\tau)}d\tau.\end{aligned}\tag{35}$$

From eq. (35), we obtain $\|u(\varepsilon)\|_{\Gamma_\nabla(\varepsilon)}\to0$ as $\|(\phi_2,\varphi_2)-(\phi_1,\varphi_1)\|_{\Gamma_\nabla(\varepsilon)}\to0$. Therefore, $\mathscr{X}_1$ is continuous on $\Gamma_\nabla(\varepsilon)$. Similar logic suggests that $\mathscr{X}_2$ is continuous. ■

Where $\Gamma_\nabla(\varepsilon)$ is closed and convex, and by Lemmas 6 and 7, Schauder's fixed point theorem, the following result is valid.

Theorem 8. *$\mathscr{X}$ admits at least one fixed point $(s^*_\nabla(\varepsilon),i^*_\nabla(\varepsilon))\in\Gamma_\nabla(\varepsilon)$.*

We present some previous estimates for the fixed point $(s^*_\nabla(\varepsilon),i^*_\nabla(\varepsilon))$ for $\mathscr{X}$ in $C^{1,1}([-\nabla,\nabla],\mathbb{R}^2)$, where

$$C^{1,1}([-\nabla,\nabla])=\{u\in C^1([-\nabla,\nabla],\mathbb{R}^2):u\text{ and }u'\text{ are Lipschitz continuous}\},$$

endowed with the norm

$$\|u\|_{C^{1,1}([-\nabla,\nabla])}=\max_{x\in[-\nabla,\nabla]}|u(x)|+\max_{x\in[-\nabla,\nabla]}|u'(x)|+\sup_{x,y\in[-\nabla,\nabla],x\ne y}\frac{|u'(x)-u'(y)|}{|x-y|}.\tag{36}$$

Next, we have the outcomes shown below.

Lemma 9. *Put $(s^*_\nabla(\varepsilon),i^*_\nabla(\varepsilon))$ be the fixed point of map $\mathscr{F}$, therefore there is a constant $C>0$ independent of ∇ satisfying $\|s^*_\nabla(\varepsilon)\|_{C^{1,1}([-\nabla,\nabla])}\le C$ and $\|i^*_\nabla(\varepsilon)\|_{C^{1,1}([-\nabla,\nabla])}\le C$, $\forall\nabla>\max\{|\varepsilon^*|,|\varepsilon_0|,r\}$.*

Proof. Obviously, we have

$$\begin{aligned}\zeta s^{*'}_\nabla(\varepsilon)&=d_1J*s^*_\nabla(\varepsilon)-d_1s^*_\nabla(\varepsilon)+\Lambda-(\mu+\gamma_C)s^*_\nabla(\varepsilon)-L(s^*_\nabla(\varepsilon),i^*_\nabla(\varepsilon)),\\\zeta i^{*'}_\nabla(\varepsilon)&=d_2J*i^*_\nabla(\varepsilon)-(d_2+(\mu+\theta)+\delta)i^*_\nabla(\varepsilon)+L(s^*_\nabla(\varepsilon-\zeta\tau),i^*_\nabla(\varepsilon-\zeta\tau)),\end{aligned}\tag{37}$$

for $\varepsilon\in[-\nabla,\nabla]$, where

$$\hat{s}_\nabla(\varepsilon)=\begin{cases}s^*_\nabla(\nabla),\varepsilon>\nabla,\\s^*_\nabla(\varepsilon),|\varepsilon|\le\nabla,\\s^-(\nabla),\varepsilon<-\nabla,\end{cases}\quad\hat{i}_\nabla(\varepsilon)=\begin{cases}i^*_\nabla(\nabla),\varepsilon>\nabla,\\i^*_\nabla(\varepsilon),|\varepsilon|\le\nabla,\\i^-(\nabla),\varepsilon<-\nabla.\end{cases}.$$

Since $s^*_\nabla(\varepsilon) \le s^0$ and $i^*_\nabla(\varepsilon) \le \mathscr{B}$ for $\varepsilon \in [-\nabla, \nabla]$, and eq. (37), we can obtain

$$|s^{*'}_\nabla(\varepsilon)| \le \frac{1}{\zeta}(2d_1 s^0 + \Lambda + (\mu+\gamma_C)s^0 + (\frac{\partial L(s^0,0)}{\partial i})\mathscr{B}) := L_1, \tag{38}$$

$$|i^{*'}_X(\varepsilon)| \le \frac{1}{\zeta}(2d_2\mathscr{B} + ((\mu+\theta)+\delta)\mathscr{B} + (\frac{\partial L(s^0,0)}{\partial i})\mathscr{B}) := L_2. \tag{39}$$

Thus,

$$|s^*_\nabla(\varepsilon) - s^*_\nabla(\eta)| \le L_1|\varepsilon-\eta|, \quad |i^*_\nabla(\varepsilon) - i^*_\nabla(\eta)| \le L_2|\varepsilon-\eta|. \tag{40}$$

By eqs. (37) to (39), we further have

$$\begin{aligned}\zeta|s^{*'}_\nabla(\varepsilon) - s^{*'}_\nabla(\eta)| \le{}& d_1\int_{-\infty}^{+\infty} J(y)(\hat{s}_\nabla(\varepsilon-y) - \hat{s}_\nabla(\eta-y))\,dy + (d_1+(\mu+\gamma_C))|s^*_\nabla(\varepsilon) - s^*_\nabla(\eta)| \\ &+ \left|L(s^*_\nabla(\varepsilon), i^*_\nabla(\varepsilon)) - L(s^*_\nabla(\eta), i^*_\nabla(\eta))\right|, \end{aligned} \tag{41}$$

$$\begin{aligned}\zeta|i^{*'}_\nabla(\varepsilon) - i^{*'}_\nabla(\eta)| \le{}& d_2\int_{-\infty}^{+\infty} J(y)(\hat{i}_\nabla(\varepsilon-y) - \hat{i}_\nabla(\eta-y))\,dy + (d_2+(\mu+\theta)+\delta)|i^*_\nabla(\varepsilon) - i^*_\nabla(\eta)| \\ &+ \left|L(s^*_\nabla(\varepsilon-\zeta\tau), i^*_\nabla(\varepsilon-\zeta\tau)) - L(s^*_\nabla(\eta-\zeta\tau), i^*_\nabla(\eta-\zeta\tau))\right|. \end{aligned} \tag{42}$$

Let $[-r,r]$ denote the compact support of $J(x)$. Since $J(x)$ is a C^1- function, there is a constant $L_J, r > 0$ that confirms $J(x) \le L_J$. So $|J_n(x_1) - J_n(x_2)| \le L_J|x_1 - x_2|,1\ \forall x_1, x_2 \in [-r,r]$. Thus, we infer that

$$\begin{aligned}\int_{-\infty}^{+\infty} J(y)\hat{s}_\nabla(\varepsilon-y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{s}_\nabla(\eta-y)\,dy ={}& \int_{\eta-r}^{\varepsilon-r} J(y)s_\nabla(y)\,dy + \int_{\varepsilon+r}^{\eta+r} J(y)s_\nabla(y)\,dy \\ &+ \int_{\eta-r}^{\varepsilon+r}(J(y-\eta) - J(y-\varepsilon))s_\nabla(y)\,dy, \\ \le{}& 4L_J r s^0|\varepsilon-\eta|. \end{aligned}$$

Likewise, we obtain

$$\int_{-\infty}^{+\infty} J(y)\hat{i}_\nabla(\varepsilon-y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{i}_\nabla(\eta-y)\,dy \le 4L_J r\mathscr{B}|\varepsilon-\eta|.$$

Then, it follows from eqs. (34) and (40) that

$$\left|L(s^*_\nabla(\varepsilon-\zeta\tau), i^*_\nabla(\varepsilon-\zeta\tau)) - L(s^*_\nabla(\eta-\zeta\tau), i^*_\nabla(\eta-\zeta\tau))\right| \le M_4(L_1+L_2)|\varepsilon-\eta|. \tag{43}$$

Combining eqs. (41) to (43), we know

$$|s^{*'}_\nabla(\varepsilon) - s^{*'}_\nabla(\eta)| \le C_s|\varepsilon-\eta| \quad \text{and} \quad |i^{*'}_\nabla(\varepsilon) - i^{*'}_\nabla(\eta)| \le C_i|\varepsilon-\eta|,$$

where

$$C_s = \frac{1}{\zeta}(4d_1L_J r s^0 + (d_1+(\mu+\gamma_C))L_1 + M_4(L_1+L_2)),$$

$$C_i = \frac{1}{\zeta}(4d_2L_J r\mathscr{B} + (d_2+(\mu+\theta)+\delta)L_2 + M_4(L_1+L_2)).$$

Consequently, we have that

$$\|s^*_\nabla(\varepsilon)\|_{C^{1,\varepsilon}([-\nabla,\nabla])} \le C,$$

$$\|i^*_\nabla(\varepsilon)\|_{C^{1,\varepsilon}([-\nabla,\nabla])} \le C_i,$$

where $C = \max\{s^0 + L_1 + C_s, \mathscr{B} + L_2 + C_i\}$. ■

4.3. Existence of a noncritical traveling wave solution

Theorem 10. *Let $R_0 > 1$ and $\zeta > \zeta^*$, then eq. (5) has a solution $(s^*(\varepsilon), i^*(\varepsilon))$ defined for $\varepsilon \in \mathbb{R}$ satisfying $s^-(\varepsilon) \le s^*(\varepsilon) \le s^0$, $i^-(\varepsilon) \le i^*(\varepsilon) \le i^+(\varepsilon)$ for $\varepsilon \in \mathbb{R}$.*

Proof. $\{X_n\}_{n=1}^{\infty}$, with $X_n > \max\{|\varepsilon^*|, |\varepsilon_0|, r\}$ and $\lim_{n\to\infty} X_n = +\infty$. Schauder's fixed point theorem states that a fixed point $(s^*_{X_n}(\varepsilon), i^*_{X_n}(\varepsilon)) \in \Gamma_{X_n}(z)$ of the map $\mathscr{F}$ exists for every X_n. Lemma 9 implies that $\|s^*_{X_n}(\varepsilon)\|_{\mathbb{C}[1-\alpha, X_n, X_n]} \le C_i$ and $\|i^*_{X_n}(\varepsilon)\|_{\mathbb{C}[1-\alpha, X_n, X_n]} \le C$, $n = 1, 2, \ldots$. For any integer k, $\{(s^*_{X_n}(\varepsilon) i^*_{X_n}(\varepsilon))\}$ and $\{(s^{*'}_{X_n}(\varepsilon), i^{*'}_{X_n}(\varepsilon))\}$, with $n \ge k$, are uniformly bounded and equicontinuous on $[-X_k, X_k]$. The Arzelà-Ascoli theorem and the diagonal extraction strategy guarantee that a subsequence $\{(s^*_{X_m}(\varepsilon), i^*_{X_m}(\varepsilon))\}$ converges uniformly in each $[-X_k, X_k]$, $(k = 1, 2, \ldots)$, as $m \to \infty$.
Assuming $\lim_{m\to\infty}(s^*_{X_m}(\varepsilon), i^*_{X_m}(\varepsilon)) = (s^*(\varepsilon), i^*(\varepsilon))$, we have

$$\lim_{m\to\infty} (s^{*'}_{X_m}(\varepsilon), i^{*'}_{X_m}(\varepsilon)) = (s^{*'}(\varepsilon), i^{*'}(\varepsilon)).$$

Let r be the supported radius of $J(\varepsilon)$. Since $(s^*_{X_m}(\varepsilon), i^*_{X_m}(\varepsilon)) \le (s^{+*}(\varepsilon), i^{+*}(\varepsilon))$ for $\varepsilon \in \mathbb{R}$ and $m = 1, 2, \ldots$, using the Lebesgue dominated convergence theorem, it follows that

$$\lim_{m\to\infty} \int_{\mathbb{R}} J(\varepsilon) s^*_{X_m}(\varepsilon - y) d\varepsilon = \lim_{m\to\infty} \int_{-r}^{r} J(\varepsilon) s^*_{X_m}(\varepsilon - y) d\varepsilon = J * s^*(\varepsilon).$$

By the same way, we can obtain $\lim_{m\to\infty} i^*_{X_m}(\varepsilon) = J * i^*(\varepsilon)$. Then, $(s^*(\varepsilon), i^*(\varepsilon))$ satisfies eq. (5)) and $s^-(\varepsilon) \le s^*(\varepsilon) \le s^0$ and $i^-(\varepsilon) \le i^*(\varepsilon) \le i^+(\varepsilon)$ for $\varepsilon \in \mathbb{R}$.

Now, we prove $s^0 > s^*(\varepsilon) > 0$ and $i^*(\varepsilon) > 0$. Since $s(-\infty) = s^0 > 0$, assume that there is $\varepsilon_{00} \in \mathbb{R}$ verifying $s(\varepsilon_{00}) = 0$ and $s(\varepsilon) > 0$, $\forall \varepsilon \in (-\infty, \varepsilon_{00})$, then $s'(\varepsilon_{00}) \le 0$. The first equation of eq. (5) gives

$$d_1 \int_{-\infty}^{+\infty} J(y) s(\varepsilon_{00} \breve{} y) dy + \Lambda \le 0.$$

This is a contradiction. Thus, $s^*(\varepsilon) > 0$, $\forall \varepsilon \in \mathbb{R}$. Likewise, we obtain $i^*(\varepsilon) > 0$, $\forall \varepsilon \in \mathbb{R}$. Now, we prove $s^*(\varepsilon) < s^0$. Assuming that there is $\varepsilon_{00} \in \mathbb{R}$ satisfying $s^*(\varepsilon_{00}) = s^0$, then, $s^{*'}(\varepsilon_{00}) \ge 0$. Together with the first equation of eq. (5) yields

$$d_1 \int_{-\infty}^{+\infty} J(y)(s(\varepsilon_{00} - y) - s^0) dy + \Lambda - (\mu + \gamma_C) s^0 - L(s^*(\varepsilon_{00}), i^*(\varepsilon_{00})) \ge 0,$$

that is,

$$d_1 \int_{-\infty}^{+\infty} J(y)(s(\varepsilon_{00} - y) - s^0) dy L(s^*(\varepsilon_{00}), i^*(\varepsilon_{00})) \ge 0,$$

this is a contradiction with $s^*(\varepsilon_{00} \breve{} y) \breve{} s^0 \le 0$ and $L(s^*(\varepsilon_{00}), i^*(\varepsilon_{00})) > 0$. Thus, $s^*(\varepsilon) < s^0$, $\forall \varepsilon \in \mathbb{R}$. ■

Theorem 11. *Let $R_0 > 1$ and $\zeta > \zeta^*$, then (5) has a solution $(s^*(\varepsilon), i^*(\varepsilon))$ defined for $\varepsilon \in \mathbb{R}$ satisfying $\lim_{\varepsilon\to\infty}(s^*(\varepsilon), i^*(\varepsilon)) = (s^0, 0)$, $0 < s^*(\varepsilon) \le s^0$, and $i^*(\varepsilon) > 0$ for $\varepsilon \in \mathbb{R}$.*

Proof. By Theorem 10, there is a solution sequence $\Phi_n(\varepsilon) = (s^*_n(\varepsilon), i^*_n(\varepsilon))$, $n \in \mathbb{N}^*$ and $\varepsilon \in \mathbb{R}$, verifying

$$\begin{aligned} \zeta s^{*'}_n(\varepsilon) &= d_1 J * s^*_n(\varepsilon) - d_1 s^*_n(\varepsilon) + \Lambda - (\mu + \gamma_C) s^*_n(\varepsilon) - L(s^*_n(\varepsilon), i^*_n(\varepsilon)), \\ \zeta i^{*'}_n(\varepsilon) &= d_2 J * i^*_n(\varepsilon) - (d_2 + \mu + \theta + \delta) i^*_n(\varepsilon) + L(s^*_n(\varepsilon - \zeta\tau), i^*_n(\varepsilon - \zeta\tau)), \end{aligned} \tag{44}$$

and

$$s(\varepsilon) < s^*_n(\varepsilon) \le s^0, \quad i(\varepsilon) \le i^*_n(\varepsilon) \le i(\varepsilon), \quad s^*_n(\varepsilon) > 0, \quad i^*_n(\varepsilon) > 0, \quad \varepsilon \in \mathbb{R}.$$

In $[-k-1, k-1]$, we select subsequences $\{\Phi_{k-1,m}(\varepsilon)\}$ of $\{\Phi_{k-2,m}(\varepsilon)\}$ satisfying $\{\Phi_{k-1,m}(\varepsilon)\}$ and $\{\Phi'_{k-1,m}(\varepsilon)\}$ converge uniformly on $[-k-1, k-1]$ when $m \to \infty$. We also have $\mathscr{B}_{k-1,m} \le e^{1+\varepsilon}$ for all $\varepsilon \in [-k-1, k-1]$.

As $\mathscr{B}_{k-1,m}$ is uniformly restricted on $[-k,k]$, we may argue that $i^*_{k,m}(\varepsilon) \le e^{1+\varepsilon}$ for all $\varepsilon \in [-k,k]$. Thus, for $m > m_k$, $\{\Phi_{k,m}(\varepsilon)\}$ is uniformly restricted on $[-k,k]$.

The proof of Lemma 7 shows that both $\{\Phi_{k-1,m}(\varepsilon)\}$ and $\{\Phi'_{k-1,m}(\varepsilon)\}$ are continuous and uniformly bounded on $[-k,k]$. $\{\Phi_{k,m}(\varepsilon)\}$ is a subsequence of $\{\Phi_{k-1,m}(\varepsilon)\}$ that converges uniformly on $[-k,k]$ for both $\{\Phi_{k,m}(\varepsilon)\}$. as $m \to \infty$. Moreover, $i^*_{k,m}(\varepsilon) \le e^{\lambda_1 \varepsilon}$, $\forall \varepsilon \in [-k,k]$. The diagonal extraction approach implies that there are subsequences $\{\Phi_{m,m}(\varepsilon)\}$ and $\{\Phi'_{m,m}(\varepsilon)\}$ that converge uniformly on each $[-k,k]$ (k = 1, 2, 3,...). Consider $\{\Phi_{m,m}(\varepsilon)\} \to (s^*(\varepsilon), i^*(\varepsilon))$ as $m \to +\infty$. Thus, $\{\Phi'_{m,m}(\varepsilon)\} \to (s^{*'}(\varepsilon), i^{*'}(\varepsilon))$ and $m \to +\infty$. Because for each $m \in \mathbb{N}^*$, we have

$$\begin{aligned} \zeta s^{*'}_{m,m}(\varepsilon) &= d_1 J * s^*_{m,m}(\varepsilon) - d_1 s^*_{m,m}(\varepsilon) + \Lambda - (\mu + \gamma_C) s^*_{m,m}(\varepsilon) - L(s^*_{m,m}(\varepsilon), i^*_{m,m}(\varepsilon)), \\ \zeta i^{*'}_{m,m}(\varepsilon) &= d_2 J * i^*_{m,m}(\varepsilon) - (d_2 + \mu + \theta + \delta) i^*_{m,m}(\varepsilon) + L(s^*_{m,m}(\varepsilon - \zeta\tau), i^*_{m,m}(\varepsilon - \zeta\tau)), \end{aligned} \tag{45}$$

when $m \to +\infty$, utilising the continuity of $L(s(\varepsilon), i(\varepsilon))$ function and the dominated convergence theorem yields

$$\begin{aligned} \zeta s^{*'}(\varepsilon) &= d_1 J * s^*(\varepsilon) - d_1 s^*(\varepsilon) + \Lambda - (\mu + \gamma_C) s^*(\varepsilon) - L(s^*(\varepsilon), i^*(\varepsilon)), \\ \zeta i^{*'}(\varepsilon) &= d_2 J * i^*(\varepsilon) - (d_2 + \mu + \theta + \delta) i^*(\varepsilon) + L(s^*(\varepsilon - \zeta\tau), i^*(\varepsilon - \zeta\tau)). \end{aligned} \tag{46}$$

For every $\varepsilon \in \mathbb{R}$. That is, $(s^*(\varepsilon), i^*(\varepsilon))$ is the solution of eq. (5) for $\varepsilon \in \mathbb{R}$. eq. (44) gives $s(\varepsilon) < s^*(\varepsilon) \le s^0$ and $i(\varepsilon) \le i^*(\varepsilon)$ for $\varepsilon \in \mathbb{R}$. For any integers $k > 0$ and $m \ge k$, $i^*_{m,m}(\varepsilon) \le e^{\lambda_1 \varepsilon}$, $\forall \varepsilon \in [-k,k]$, we further get $i^*(\varepsilon) \le e^{\lambda_1 \varepsilon}$, $\varepsilon \in \mathbb{R}$. The upper-lower solutions show that $(s^*(\varepsilon), i^*(\varepsilon))$ satisfies $\lim_{\varepsilon \to -\infty}(s^*(\varepsilon), i^*(\varepsilon)) = (s^0, 0)$. Similar to Theorem 10, we obtain $0 < s^*(\varepsilon) < s^0$ for every $\varepsilon \in \mathbb{R}$. Assume $\varepsilon' \in \mathbb{R}$, $i^*(\varepsilon') = 0$, $i^*(\varepsilon) > 0$, and $\forall \varepsilon \in (-\infty, \varepsilon')$. Clearly, for $\varepsilon' > \varepsilon_0$, $i^{*'}(\varepsilon') \le 0$. Using the second equation of eq. (46), we obtain

$$\zeta i^{*'}(\varepsilon') = d_2 J * i^*(\varepsilon') - (d_2 + (\mu + \theta + \delta) i^*(\varepsilon') + L(s^*(\varepsilon' - \zeta\tau), i^*(\varepsilon' - \zeta\tau)) > 0.$$

This is a contradiction. Then, $i^*(\varepsilon) > 0$, $\forall \varepsilon \in \mathbb{R}$. ■

Let's define $(s^*(\varepsilon), i^*(\varepsilon))$ as in Theorem 11. To calculate the asymptotic boundary condition $(s^*(\varepsilon), i^*(\varepsilon)) \to (s^*, i^*)$ as $\varepsilon \to +\infty$. To establish the existence of non-critical TWS, we need to demonstrate that $(s^*(\varepsilon), i^*(\varepsilon)) \to (s^*, i^*)$ as $\varepsilon \to \infty$ using the Lyapunov-LaSalle theorem. The findings are emphasised as follows:

Lemma 12. *$(s^*(\varepsilon), i^*(\varepsilon)) \to (s^*, i^*)$ uniformly as $\varepsilon \to +\infty$.*

Proof. We construct the Lyapunov functional.

$$V(\varepsilon) = V_1(\varepsilon) + V_2(\varepsilon), \tag{47}$$

where

$$\begin{aligned} V_1(\varepsilon) &= \zeta\left(s(\varepsilon) - s^* - \int_{s^*}^{s(\varepsilon)} \frac{L(s^*, i^*)}{L(\xi, i^*)} d\xi + i^* h\left(\frac{i(\varepsilon)}{i^*}\right)\right) + d_1 s^* K_1(\varepsilon) + d_2 K_2(\varepsilon), \\ V_2(\varepsilon) &= L(s^*, i^*) \int_0^{\zeta\tau} h\left(\frac{L(s(\varepsilon - \varepsilon), i(\varepsilon - \varepsilon))}{L(s^*, i^*)}\right) d\varepsilon. \end{aligned}$$

with $h(x) = x - 1 - \ln(x)$; $x \in \mathbb{R}^+$, clearly $h(x) > 0$ for all $x > 0$. Then,

$$\begin{aligned} K_1(\varepsilon) = \int_0^{+\infty} a^+(y)\left[h\left(\frac{s(\varepsilon - y)}{s^*}\right) - h\left(\frac{L(s^*, i^*) s(\varepsilon - y)}{L(s(\varepsilon), i^*) s^*}\right)\right] dy - \int_{-\infty}^0 a^-(y)\left[h\left(\frac{s(\varepsilon - y)}{s^*}\right)\right. \\ \left. - h\left(\frac{L(s^*, i^*) s(\varepsilon - y)}{L(s(\varepsilon), i^*) s^*}\right)\right] dy, \end{aligned}$$

$$K_2(\varepsilon) = \int_0^{+\infty} b^+ h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy - \int_{-\infty}^0 b^- h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy.$$

Thanks to [5, Theorem 1], $s(\varepsilon)>0, i(\varepsilon)>0$, we can have that $K_1(\varepsilon), K_2(\varepsilon)$ are bounded from below. Thus $V(s,i)(\varepsilon)$ is well defined and bounded from below. Noting that $a^\pm$ satisfies $a^\pm(0)=\frac{1}{2}$, $\frac{da^+(y)}{dy}=J(y)$ and $\frac{da^-(y)}{dy}=-J(y)$, we have

$$\begin{aligned}
\frac{dK_1(\varepsilon)}{d\varepsilon} &= \frac{d}{d\varepsilon}\int_0^{+\infty} a^+(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy \\
&\quad -\frac{d}{d\varepsilon}\int_{-\infty}^0 a^-(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy, \\
&= \int_0^{+\infty} a^+(y)\frac{d}{d\varepsilon}\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy \\
&\quad -\int_{-\infty}^0 a^-(y)\frac{d}{d\varepsilon}\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy, \\
&= -\int_0^{+\infty} a^+(y)\frac{d}{dy}\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy \\
&\quad +\int_{-\infty}^0 a^-(y)\frac{d}{dy}\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy, \\
&= h\left(\frac{s(\varepsilon)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)-\int_{-\infty}^{+\infty} J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy.
\end{aligned}$$

Similarly,

$$\frac{dK_2(\varepsilon)}{d\varepsilon} = h\left(\frac{i}{i^*}\right) - \int_{-\infty}^{+\infty} J(y)h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy.$$

Now, we compute $\frac{dV_2(\varepsilon)}{d\varepsilon}$

$$\begin{aligned}
\frac{dV_2(\varepsilon)}{d\varepsilon} &= \frac{d}{d\varepsilon}\left(L(s^*,i^*)\int_0^{\zeta\tau} h\left(\frac{L(s(\varepsilon-\varepsilon),i(\varepsilon-\varepsilon))}{L(s^*,i^*)}\right)d\varepsilon\right), \\
&= L(s^*,i^*)\int_0^{\zeta\tau}\frac{d}{d\varepsilon}\left(h\left(\frac{L(s(\varepsilon-\varepsilon),i(\varepsilon-\varepsilon))}{L(s^*,i^*)}\right)d\varepsilon\right), \\
&= -L(s^*,i^*)\int_0^{\zeta\tau}\frac{d}{d\varepsilon}\left(h\left(\frac{L(s(\varepsilon-\varepsilon),i(\varepsilon-\varepsilon))}{L(s^*,i^*)}\right)d\varepsilon\right), \\
&= -L(s^*,i^*)\left(h\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))}{L(s^*,i^*)}\right)-h\left(\frac{L(s(\varepsilon),i(\varepsilon))}{L(s^*,i^*)}\right)\right), \\
&= -L(s^*,i^*)\left[\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))}{L(s^*,i^*)}\right)-1-\ln\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))}{L(s^*,i^*)}\right)-\left(\frac{L(s(\varepsilon),i(\varepsilon))}{L(s^*,i^*)}\right)+1\right. \\
&\quad \left.+\ln\left(\frac{L(s(\varepsilon),i(\varepsilon))}{L(s^*,i^*)}\right)\right].
\end{aligned}$$

First, differentiating $V_1(\varepsilon)$ and using the chain rule, we obtain

$$\frac{d}{d\varepsilon}\left(s(\varepsilon)-s^*-\int_{s^*}^{s(\varepsilon)}\frac{L(s^*,i^*)}{L(\xi,i^*)}d\xi\right) = s'(\varepsilon)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right).$$

Similarly, since $h'(x)=1-\frac{1}{x}$, we have

$$\frac{d}{d\varepsilon}h\left(\frac{i(\varepsilon)}{i^*}\right) = \frac{i'(\varepsilon)}{i^*}\left(1-\frac{i^*}{i(\varepsilon)}\right).$$

Note that (s^*, i^*) satisfies

$$\Lambda = (\mu+\gamma_C)s^* + L(s^*,i^*),$$
$$(\mu+\delta+\theta)i^* = L(s^* i^*).$$

Then, we obtain

$$\begin{aligned}\frac{dV_1(\varepsilon)}{d\varepsilon} &= \left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)\left(d_1(J*s(\varepsilon)-s(\varepsilon))+\mu s^*+L(s^*,i^*)-(\mu+\gamma_C)s(\varepsilon)-L(s(\varepsilon),i(\varepsilon))\right)\\ &\quad+\left(1-\frac{i^*}{i(\varepsilon)}\right)\Bigg(d_2(J*i(\varepsilon)-i(\varepsilon))+L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))-L(s^*,i^*)\frac{i(\varepsilon)}{i^*}\\ &\quad+L(s(\varepsilon),i(\varepsilon))-L(s(\varepsilon),i(\varepsilon))\Bigg)+h\left(\frac{s(\varepsilon)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\\ &\quad-\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy\\ &\quad+h\left(\frac{i(\varepsilon)}{i^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy,\\ &= O_1(\varepsilon)+O_2(\varepsilon).\end{aligned}$$

We put

$$\begin{aligned}O_1(\varepsilon) &= \left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)(d_1(J*s(\varepsilon)-s(\varepsilon)))+d_1s^*\Bigg((h\left(\frac{s(\varepsilon)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\\ &\quad-\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy\Bigg)+\left(1-\frac{i^*}{i(\varepsilon)}\right)(d_2(J*i(\varepsilon)-i(\varepsilon)))\\ &\quad+d_2i^*\left(h\left(\frac{i(\varepsilon)}{i^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy\right).\end{aligned}$$

For O_1, using $\ln\left(\frac{s}{s^*}\right)=\ln\left(\frac{s(\varepsilon-y)}{s^*}\right)-\ln\left(\frac{s(\varepsilon-y)}{s(\varepsilon)}\right)$ and $\ln\left(\frac{i}{i^*}\right)=\ln\left(\frac{i(\varepsilon-y)}{i^*}\right)-\ln\left(\frac{i(\varepsilon-y)}{i(\varepsilon)}\right)$,

$$\begin{aligned}\mathscr{P}_1 &= d_1s^*\int_{-\infty}^{+\infty}J(y)\Bigg[\frac{s(\varepsilon-y)}{s^*}-\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}-\ln\left(\frac{s(\varepsilon)}{s^*}\right)+\ln\left(\frac{L(s^*,i^*)s(\varepsilon)}{L(s(\varepsilon),i^*)s^*}\right)+\ln\left(\frac{s(\varepsilon-y)}{s(\varepsilon)}\right)\\ &\quad-\ln\left(\frac{s(\varepsilon-y)}{s(\varepsilon)}\right)+1-1\Bigg]dy-d_1s^*\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy,\\ &= d_1s^*\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}+1+\ln\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s*}\right)\right]dy\\ &\quad-d_1s^*\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy,\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathscr{P}_1 &= \left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)(d_1(J*s(\varepsilon)-s(\varepsilon)))+d_1s^*\Bigg(h\left(\frac{s(z)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\\ &\quad-\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{s(\varepsilon-y)}{s^*}\right)-h\left(\frac{L(s^*,i^*)s(\varepsilon-y)}{L(s(\varepsilon),i^*)s^*}\right)\right]dy\Bigg),\end{aligned}$$

$$\begin{aligned}\mathscr{P}_2 &= d_2i^*\int_{-\infty}^{+\infty}J(y)\left[\frac{i(\varepsilon-y)}{i^*}-\frac{i(\varepsilon-y)}{i(\varepsilon)}-\ln(\frac{s(\varepsilon)}{s^*})\right]dy-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy,\\ &= d_2i^*\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{i(\varepsilon-y)}{i^*}\right)-h\left(\frac{i(\varepsilon-y)}{i(\varepsilon)}\right)\right]dy-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\varepsilon-y)}{i^*}\right)dy,\end{aligned}$$

$$
\begin{aligned}
&= -d_2 i^* \int_{-\infty}^{+\infty} J(y) h\left(\frac{i(\varepsilon-y)}{i(\varepsilon)}\right) dy, \\
\mathscr{P}_2 &= \left(1-\frac{i^*}{i(\varepsilon)}\right)\left(d_2(J * i(\varepsilon)-i(\varepsilon))\right)+d_2 i^*\left(h\left(\frac{i(\varepsilon)}{i^*}\right)-\int_{-\infty}^{+\infty} J(y) h\left(\frac{i(\varepsilon-y)}{i^*}\right) dy\right),
\end{aligned}
$$

and

$$
\begin{aligned}
O_2(\varepsilon) =& (\mu+\gamma_C) s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right) \\
&+L(s^*,i^*)\left(2-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}-\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}+\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}-\frac{i(\varepsilon)}{i^*}+\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))}{L(s^*,i^*)}\right. \\
&-\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i(\varepsilon)}-\frac{L(s(\varepsilon),i(\varepsilon))}{L(s^*,i^*)}+\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}-\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))}{L(s^*,i^*)}\right) \\
&+1+\ln\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))}{L(s^*,i^*)}\right)+\left(\frac{L(s(\varepsilon),i(\varepsilon))}{L(s^*,i^*)}\right)-1 \\
&\left.-\ln\left(\frac{L(s(\varepsilon),i(\varepsilon))}{L(s^*,i^*)}\right)+\ln\left(\frac{i^*}{i(\varepsilon)}\right)-\ln\left(\frac{i^*}{i(\varepsilon)}\right)\right), \\
=& (\mu+\gamma_C) s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right) \\
&+L(s^*,i^*)\left(2-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}-\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}+\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}-\frac{i(\varepsilon)}{i^*}-\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i}\right. \\
&\left.+\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}+1+\ln\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i(\varepsilon)}\right)-1-\ln\left(\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}\right)\right), \\
=& (\mu+\gamma_C) s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right) \\
&+L(s^*,i^*)\left(2-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}-\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}+\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}-\frac{i(\varepsilon)}{i^*}-h\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i}\right)\right. \\
&\left.+h\left(\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}\right)+\frac{L(s(\varepsilon),i^*)i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}-\frac{L(s(\varepsilon),i^*))i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}+1-1\right), \\
=& (\mu+\gamma_C) s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right) \\
&+L(s^*,i^*)\left(3-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}-\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}-\frac{L(s(\varepsilon),i^*)i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}+\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}-\frac{i(\varepsilon)}{i^*}\right. \\
&\left.-h\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i(\varepsilon)}\right)+h\left(\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}\right)+\frac{L(s(\varepsilon),i^*))i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}-1\right), \\
=& (\mu+\gamma_C) s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right) \\
&+L(s^*,i^*)\left(-h\left(\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)-h\left(\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}\right)\right. \\
&-h\left(\frac{L(s(\varepsilon),i^*)i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}\right)-h\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i}\right)+h\left(\frac{L(s(\varepsilon),i(\varepsilon))i^*}{L(s^*,i^*)i(\varepsilon)}\right) \\
&\left.+\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}-\frac{i(\varepsilon)}{i^*}+\frac{L(s(\varepsilon),i^*))i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}-1\right), \\
=& (\mu+\gamma_C) s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)+L(s^*,i^*)\left(-h\left(\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)-h\left(\frac{L(s(\varepsilon),i^*)i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}\right)\right.
\end{aligned}
$$

$$-h\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i(\varepsilon)}\right)+\left(\frac{i(\varepsilon)}{i^*}-\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}\right)\left(\frac{L(s(\varepsilon),i^*)}{L(s(\varepsilon),i(\varepsilon))}-1\right)\Bigg).$$

Then we have

$$\begin{aligned}\frac{dV(\varepsilon)}{d\varepsilon}=&(\mu+\gamma_C)s^*\left(1-\frac{s(\varepsilon)}{s^*}\right)\left(1-\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)+L(s^*,i^*)\Bigg(-h\left(\frac{L(s^*,i^*)}{L(s(\varepsilon),i^*)}\right)-h\left(\frac{L(s(\varepsilon),i^*)i(\varepsilon)}{L(s(\varepsilon),i(\varepsilon))i^*}\right)\\&-h\left(\frac{L(s(\varepsilon-\zeta\tau),i(\varepsilon-\zeta\tau))i^*}{L(s^*,i^*)i(\varepsilon)}\right)+\left(\frac{i(\varepsilon)}{i^*}-\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}\right)\left(\frac{L(s(\varepsilon),i^*)}{L(s(\varepsilon),i(\varepsilon))}-1\right)\Bigg)\\&-d_2i^*\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\varepsilon-y)}{i(\varepsilon)}\right)dy.\end{aligned}$$

Remark 2. Note that V is decreasing. By **(H)** we deduce that

$$\begin{aligned}\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}&>\frac{i(\varepsilon)}{i^*}\quad\text{for}\quad 0<i(\varepsilon)<i^*,\\\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}&\le\frac{i(\varepsilon)}{i^*}\quad\text{for}\quad i(\varepsilon)\ge i^*,\end{aligned}$$

then

$$\left(\frac{i(\varepsilon)}{i^*}-\frac{L(s(\varepsilon),i(\varepsilon))}{L(s(\varepsilon),i^*)}\right)\left(\frac{L(s(\varepsilon),i^*)}{L(s(\varepsilon),i(\varepsilon))}-1\right)\le 0.$$

and

$$\begin{aligned}L(s^*,i^*)&>L(s(\varepsilon),i^*)\quad\text{for all}\quad 0<s(\varepsilon)\le s^*,\\L(s^*,i^*)&<L(s(\varepsilon),i^*)\quad\text{for all}\quad s(\varepsilon)\ge s^*.\end{aligned}$$

Thus, $\frac{dV(\varepsilon)}{d\varepsilon}\le 0$, and $\frac{dV(\varepsilon)}{d\varepsilon}=0$ where $s(\varepsilon)=s^*$, $i(\varepsilon)=i^*$. Finally, $(s,i)(\infty)=(s^*,i^*)$. By applying the Lyapunov-LaSalle theorem, it follows that the trajectory $\Psi(\varepsilon)$ converges to (s^*,i^*) as $\varepsilon\to\infty$. Thus, we have $(s,i)\to(s^*,i^*)$ as $\varepsilon\to+\infty$, completing the proof. ■

The results of Lemma 5 implies that $s^-\le s(\varepsilon)\le s^+$, $i^-\le i(\varepsilon)\le i^+$, and $(s,i)\to(s^0,0)$ as $\varepsilon\to-\infty$. Therefore, we get that the solution converge to (EE) as $(\varepsilon\to+\infty)$. As a result, the solution satisfies the conditions eq. (6). $(s(\varepsilon),i(\varepsilon))$ corresponds to the traveling wave solution of the system eq. (1). Therefore, the obtained traveling wave solution connects the disease-free equilibrium to the endemic equilibrium. Epidemiologically, this wave describes the transition zone between:

- an uninfected region ahead of the front,
- and an endemic region behind the front where infection persists.

The monotonicity of the wave profile suggests that the invasion process is smooth. It is also irreversible, as indicated by the presence of the wave. The absence of the waves when $R_0\le 1$ shows that the reproduction threshold needs to be greater than unity for the invasion to be successful. In conclusion, the mathematical solution of the wave is a realistic model of the epidemic invasion process.

5. Existence of a Critical Traveling Wave Solution

In this section, we aim to prove that eq. (5) admits a TWS for $R_0>1$ and $\zeta=\zeta^*$.

Lemma 13. *If $R_0>1$, and $\zeta=\zeta^*$. Let*

$$\begin{aligned}s^+(\varepsilon)&=s^0, & i^+&=\min\{e^{\lambda_1\varepsilon},\mathscr{B}\},\\s^-&=\max\left\{s^0-Me^{\gamma\varepsilon},0\right\}, & i^-(\varepsilon)&=\max\{e^{\lambda_1\varepsilon}(1-Je^{\eta\varepsilon}),0\}.\end{aligned}$$

We do not offer the evidence here because it is comparable to the proof of Lemma 5. To derive the presence of a TWS for $\zeta=\zeta^*$, replace ζ with ζ^* and λ_1 with λ^*, as shown in Section 4.

6. Numerical Results

6.1. Numerical scheme

To approximate traveling wave solutions of the delayed nonlocal system, we employ an explicit finite-difference method in space combined with forward Euler time discretization. Let $x_j = j\Delta x$ and $t^n = n\Delta t$. The spatial convolution term is discretized as

$$\int_{\mathbb{R}} J(x-y)u(y,t)dy \approx \sum_{k=-M}^{M} J(x_j - x_k)u_k^n \Delta x.$$

The time delay term is implemented using stored previous time layers:

$$u(x,t-\tau) \approx u_j^{n-m_\tau}, \quad m_\tau = \left\lfloor \frac{\tau}{\Delta t} \right\rfloor.$$

The discrete infected equation, for example, reads

$$\frac{I_j^{n+1} - I_j^n}{\Delta t} = d_2 \sum_{k=-M}^{M} J(x_j - x_k)I_k^n \Delta x - (\mu + \delta + \theta)I_j^n + L(S_j^{n-m_\tau}, I_j^{n-m_\tau}).$$

The spatial domain $[-L, L]$ is truncated sufficiently large so that boundary effects do not influence wave propagation. Zero-flux boundary conditions are imposed.

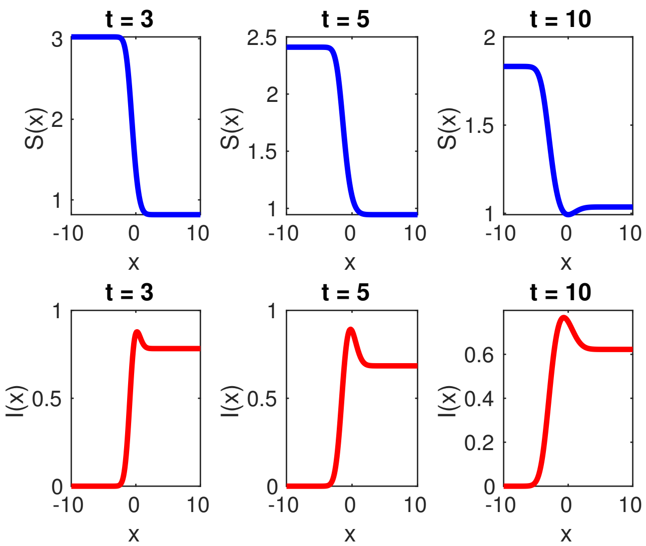

Figure 1. The curve of TWS for the system (2).

We first give numerical representations of system (1). Hence, the Figure 1 give the numerical existence of traveling waves solution and prove the theoretical results. To do this, we analyse the following beginning circumstances.

$$S_0(x) = \begin{cases} 3 & \text{if } x \in [-10, 0], \\ 0.8 & \text{if } x \in [0, 10], \end{cases},$$

$$I_0(x) = \begin{cases} 0 & \text{if } x \in [-10, 0], \\ 0.6 & \text{if } x \in [0, 10]. \end{cases}.$$

We also adopt the kernel function as

$$J(x) = Ke^{\frac{1}{x^2-1}} \; if \; -1 < x < 1 \text{ and } 0 \text{ otherwise.}$$

with $K = 0.2$ is satisfy $\int_{-1}^{1} J(x)dx = 1$. Where $L(S,I) = \frac{\psi SI}{1+\varepsilon I}$, $\Lambda = 0.5$, $\psi = 0.1$, $\mu = 0.01$, $\theta = 0.1$, $\gamma_C = 0.01$, $\rho = 0.01$ and $\varepsilon = 0.001$.

6.2. Sensitivity analysis

Now, we focus to the time delay impact on the dynamics of system (2) when we using the different values of τ. Then, the following figure shows the impact of time delay on the dynamics of system (2) according to the different values of $\tau = 0.5, 1.5, 3$ (see the Figure 2).

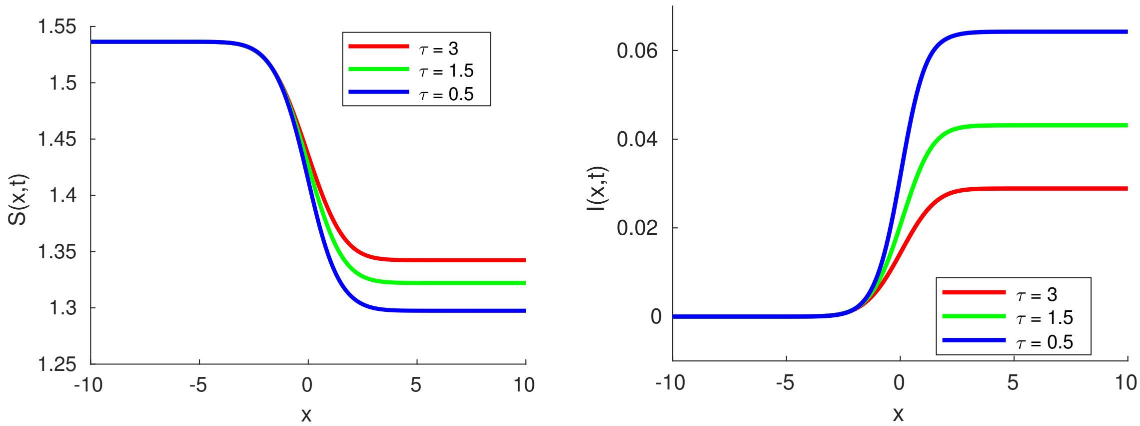

Figure 2. Effect of time delay τ on wave propagation.

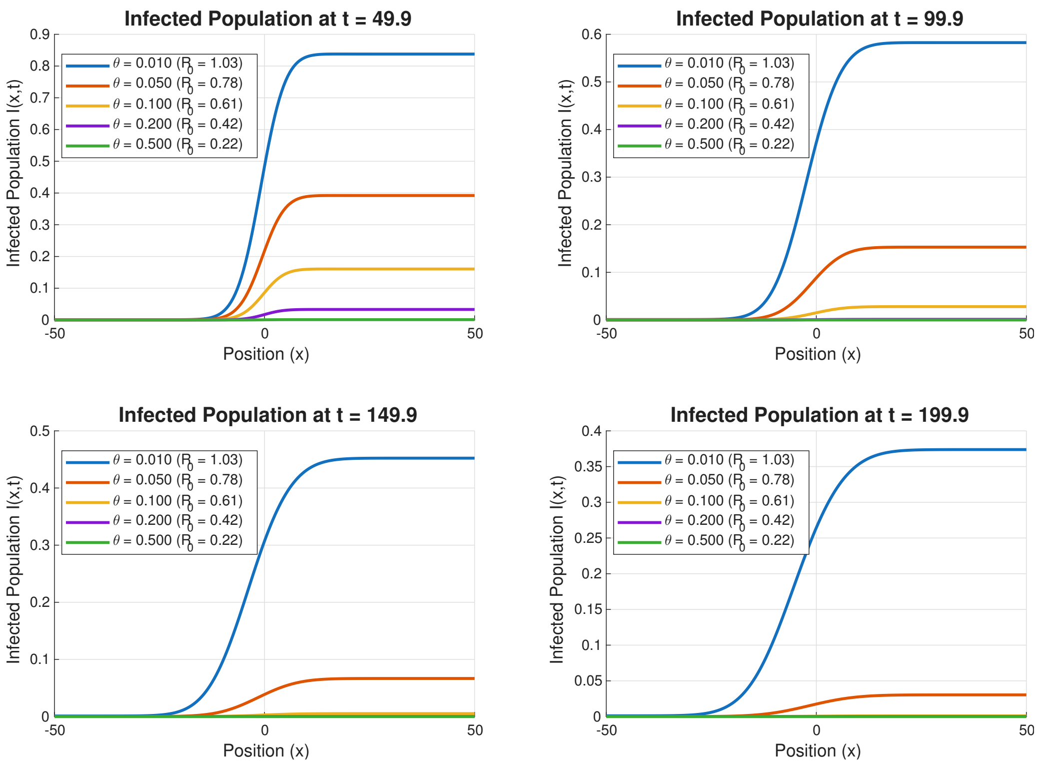

Figure 3. Effect of confinement rate (θ) on infected wave propagation.

The parameter values are chosen to ensure biologically realistic epidemic dynamics according to classical compartmental epidemic models.

- The natural death rate μ and the recovery rate δ are taken from typical ranges used in epidemic models.
- The transmission rate β is set to ensure that $R_0 > 1$ so that invasion is possible.
- The confinement parameter θ is varied to demonstrate its influence on R_0 and wave speeds where represented by the Figures 3 to 5.
- The time delay τ represents incubation time or response time and is taken from typical epidemiologically relevant ranges.
- The dispersal kernel J is taken as a Gaussian function to model short-range dispersal.

These parameter values are typical for epidemic models; see, for example, Murray [6].

6.3. Epidemiological implications

The current analysis offers several epidemiologically relevant findings: First, the confinement parameter θ has a twofold impact: It decreases the basic reproduction number and, at the same time, decreases the minimum propagation speed. This finding indicates that confinement not only prevents invasion when $R_0 < 1$, but also slows down the geographic spread of the disease when $R_0 > 1$.

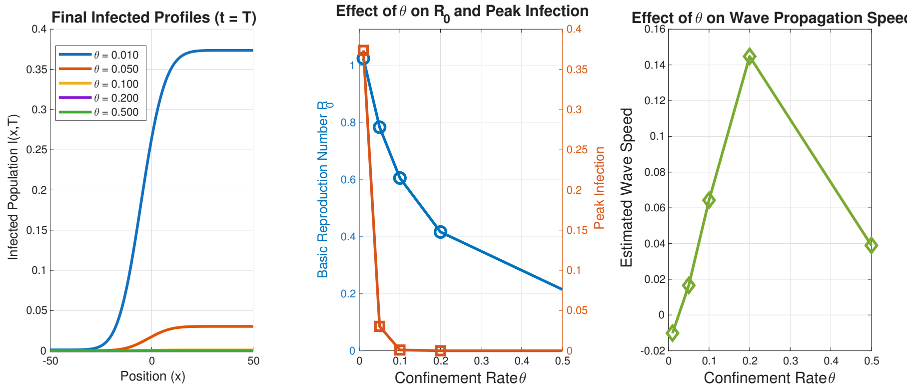

Figure 4. Sensitivity analysis of confinement rate θ.

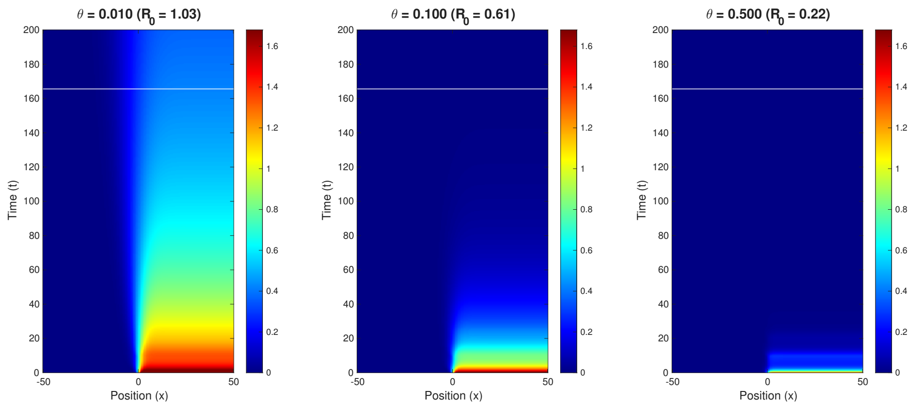

Figure 5. Effect of θ on R_0.

Second, nonlocal dispersal impacts the epidemic velocity significantly. In particular, the heavy tails of the kernel, describing the long-range human mobility, increase the minimum speed of the epidemic spread. Finally, the model introduces the impact of time delay on invasion dynamics. This impact may destabilize the equilibria and affect the epidemic spread speed, suggesting that early recognition and rapid response to the epidemic play a critical role in the epidemic spread and containment.

7. Conclusion

In this paper, we formulated and analyzed a nonlocal dispersal SICQR epidemic model with both time delay and a generalized incidence function, in addition to a confinement mechanism. The model characterizes spatial transmission through nonlocal convolution operators, temporal incubation effects through discrete delay terms, and control measures through modified transmission dynamics that account for confinement.

Our key theoretical result gives that the basic reproduction number R_0 serves as a threshold parameter for the existence of traveling wave solutions. More specifically, we established that if $R_0 > 1$ then the system admits nontrivial traveling wave solutions for any wave speed $\zeta \geq \zeta^*$, where ζ^* denotes the minimal wave speed. On the other hand, such wavefronts do not exist for speeds $\zeta < \zeta^*$, or when $R_0 \leq 1$.

The existence of traveling waves was determined by developing carefully constructed upper and lower solutions, specifically tailored to the nonlinear and delayed structure of the model. The influence of confinement, introduced via transmission terms adapted, was determined to affect both the velocity of the wave and the nature of spatial propagation. The analytic results were supplemented with numerical simulations, which exhibit spatiotemporal patterns of disease spread for various levels of confinement and values of delay.

Based on the results obtained in this paper, some promising directions for future studies can be suggested. One of the more interesting directions for extending this model could be the use of fractional derivatives. Recent studies showed that the use of fractional derivatives, for instance, the Caputo derivative or the Caputo-Fabrizio derivative, can reveal the memory effects and anomalous diffusion that can often be observed in real-life data but cannot be described by integer-order models. Such a fractional SICQR model can reveal a more complex understanding of the role of past states of infections in the current model and the role of spatial heterogeneity in traveling wave solutions. Another direction for future studies could be the validation of such a fractional model with real-life data and the use of optimal control methods in this more complex model.

Supplementary Information

Author Contributions. **Nidhal Faisal Ali:** Writing-Original draft, Methodology. **Rassim Darazirar:** Iinvestigation, Formal analysis, Writing- Original draft. **Sawsan Mohsen Abed:** Writing-review editing. **Ahmed Ali Mohsen:** Supervision, review. **Ebenezer Bonyah:** Software, Supervision.

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