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RESEARCH ARTICLE

Optimal control of HIV transmission model with pre-ART counselling and treatment

Mohamad Syafi'i[1,2], **Fatmawati**[3,✉], **Ahmadin**[3], **Chidozie Williams Chukwu**[4]

[1]Doctoral of Study Program of Mathematics and Natural Sciences, Faculty of Science and Technology, Universitas Airlangga, Surabaya, Indonesia

[2]Department of Mathematics, Faculty of Science and Technology, Imam Bonjol State Islamic University, Padang, Indonesia

[3]Department of Mathematics, Faculty of Science and Technology, Universitas Airlangga, Surabaya, Indonesia

[4]Department of Mathematical Sciences, Georgia Southern University, Statesboro, Georgia, 30460, USA

✉Corresponding author. Email: *fatmawati@fst.unair.ac.id*

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Editorial office: Department of Mathematics, Universitas Negeri Gorontalo, Jln. Prof. Dr. Ing. B. J. Habibie, Bone Bolango 96554, Indonesia

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Abstract. *HIV attacks CD4 cells of the immune system, leading to progressive immune deficiency. Antiretroviral therapy (ART) involves the use of HIV drugs to treat HIV infection and is administered daily to slow disease progression. This paper aims to develop and analyze a mathematical model of HIV transmission that incorporates pre-antiretroviral therapy counselling and HIV treatment to reduce the number of HIV-infected individuals with high-risk behaviours for HIV transmission. A nonlinear dynamical system is constructed, and model parameters are estimated from Indonesia's annual HIV case data using a genetic algorithm method. The model exhibits two equilibrium points: the disease-free equilibrium and the endemic equilibrium. Stability analysis shows that disease-free equilibrium is globally asymptotically stable when the basic reproduction number is less than one. Optimal control theory is applied to a system that consists of two time-dependent controls, pre-antiretroviral therapy counselling and HIV treatment. Healthcare professionals provide pre-antiretroviral therapy counselling to help people with HIV understand the disease and the benefits of antiretroviral therapy. Pontryagin's maximum principle is employed to derive optimal control conditions. The optimal control problem is numerically solved using the forward–backward sweep method with a fourth-order Runge–Kutta scheme. Three potential strategies were developed and investigated in our simulation. Implementing the two combined controls could significantly reduce the number of HIV-infected individuals and improve overall disease control in the population.*

1. Introduction

Human Immunodeficiency Virus (HIV) infection continues to be a serious global public health problem. Human Immunodeficiency Virus (HIV) is a virus that damages the body's immune system [1]. Its primary target is the CD4 cell, a type of white blood cell. By attacking and destroying these cells, the virus gradually weakens the immune system. Without treatment, this condition may progress

to acquired immunodeficiency syndrome (AIDS), which represents the most severe stage of HIV infection [1]. HIV is classified as an infectious disease. Between 1987 and June 2022, a total of 478,784 HIV cases were reported in Indonesia. During the same period, 139,500 cases of AIDS were recorded. From January to June 2022, 4,010 new HIV cases were detected, with a mortality rate of 0.4%. The majority of HIV infections (70.5%) occurred in the 25–49 age group [2]. Eliminating HIV infection is part of Indonesia's sustainable development goals. Prevention strategies include abstinence, maintaining faithfulness with one partner, practicing safe sex through condom use, avoiding the sharing of contaminated needles or syringes, and undergoing routine HIV screening for individuals at high risk [3]. Antiretroviral therapy (ART) is the recommended treatment for individuals living with HIV. Although ART cannot cure HIV, it helps preserve the immune system and reduce the risk of coinfections. HIV-positive individuals undergoing ART who sustain an undetectable viral load are not at risk of transmitting the virus to sexual partners. Hence, expanding early access to ART and ensuring long-term treatment adherence represent critical measures for both curbing HIV transmission and improving the overall well-being of affected populations [1].

HIV-infected individuals are more vulnerable to coinfection. Many people living with HIV may initially have limited knowledge about the disease and its prevention. Without adequate understanding of HIV transmission and health protocols, infected individuals may engage in high-risk behaviours, which can increase the rate of HIV transmission [4]. HIV-infected individuals with high-risk behaviours are individuals living with HIV who engage in behaviours that increase the likelihood of transmitting the virus to susceptible individuals, such as inconsistent condom use, multiple sexual partners, or sharing contaminated needles. Individuals in this group also do not initiate antiretroviral therapy (ART), which can further elevate the risk of HIV transmission. Without ART, their health may progressively decline, leading to the development of AIDS, increased vulnerability to co-infections, and a higher risk of death. Common co-infections in people living with HIV include candidiasis, tuberculosis, herpes simplex virus, and pneumonia, further elevating the risk of mortality [5]. AIDS represents the most advanced and severe stage of HIV infection. At this stage, extensive damage to the immune system leaves the body unable to combat opportunistic infections. The condition is diagnosed in HIV-positive individuals whose CD4 cell count falls below 200 $\frac{cells}{mm^3}$ or who present with specific opportunistic infections. Following an AIDS diagnosis, individuals typically exhibit a high viral load and are highly infectious. Without treatment, the average life expectancy of an AIDS patient is approximately three years [3]. On the other hand, HIV-infected individuals with low-risk behaviours are individuals who are aware of their HIV status and adopt healthier lifestyles, practice safer sexual behaviours, seek regular medical care, and willingly initiate ART. If antiretroviral therapy (ART) is administered effectively and consistently, it reduces the viral load in the bloodstream, thereby preventing HIV-positive individuals from transmitting the virus to others [6].

There are several ways to raise awareness, such as health promotion, pre-ART counselling, and HIV-care community support. Pre-ART counselling by health professionals plays an important role to change HIV-infected individuals with high-risk behaviours to HIV-infected individuals with low-risk behaviours. This counselling guide people living with HIV to change their lifestyle voluntarily. The healthcare professional gives information about HIV disease, its transmission, prevention, and the benefit of ART. They also give emotional support to help people living with HIV accept their disease and start ART. Initiating ART is challenging because people living with HIV should decide on long-term ART by themselves without coercion from health professionals. Hence, HIV-infected individual will take ART voluntarily.

Mathematical models plays an important role to describe and solve real-world problems. In epidemiology, mathematical models are particularly useful for understanding patterns of disease transmission within a population [7]. The transmission of diseases can be represented through mathematical models such as those for HIV, tuberculosis (TB), and COVID-19 [8–12]. In this paper, we construct

and develop a mathematical model related to HIV transmission. Several previous studies have also proposed mathematical models of HIV transmission, including models that incorporate the impact of antiretroviral therapy (ART) on individuals living with HIV/AIDS [13–17]. Some researchers have developed the mathematical model of HIV/AIDS transmission with an optimal control strategy [18–21]. Marsudi et al. proposed an HIV transmission model incorporating education campaigns, screening, and treatment, with application of optimal control theory. Sensitivity analysis showed that interactions between susceptibles and unaware infectives strongly affect the reproduction number. Numerical results indicated that combining all three interventions is more effective than applying them individually, emphasizing the need for integrated strategies in HIV control [19]. In other research, Gurmu et al. developed a deterministic HIV/AIDS transmission model with optimal control strategies, incorporating prevention, screening, and treatment. Using optimal control theory, their study evaluates the effectiveness of single and combined interventions, with numerical results showing that a combination of prevention, screening, and treatment is more effective in reducing HIV/AIDS transmission than applying controls individually [21].

Based on previous research, we propose an HIV transmission model that explicitly incorporates pre-ART counselling and early ART initiation as time-dependent controls. Pre-ART counselling is essential to prepare patients for antiretroviral therapy (ART) by improving understanding of treatment goals, promoting adherence, reducing stigma and anxiety, and promoting healthy behaviour. It also strengthens psychosocial readiness and equips patients to manage side effects, making it a critical step for long-term treatment success. By calibrating the model with real HIV case data in Indonesia from 2012 to 2022, we provide a numerical framework to evaluate the effectiveness of intervention strategies. The integration of time-dependent controls into the mathematical model seeks to determine an optimal strategy to minimize HIV transmission, particularly by reducing the population of HIV-infected individuals with high-risk behaviours. This approach offers actionable insights for optimizing HIV prevention and treatment policies for Indonesia.

2. Formulation of a HIV Model

This section introduces the formulation of an HIV transmission model with pre-antiretroviral therapy counselling. The total population is divided into four compartments: susceptible individuals (S), HIV-infected individuals who know their HIV status and engage in high-risk behaviours that contribute to HIV transmission (I_{hu}), HIV-infected individuals with low-risk behaviours for HIV transmission but have not yet initiated antiretroviral treatment (I_{ha}), and HIV-infected individuals on ART therapy (T_H). Hence, the total population at any time t represented by $N(t)$, with

$$N(t) = S(t) + I_{hu}(t) + I_{ha}(t) + T_H(t). \tag{1}$$

The assumptions of the HIV model to be constructed are as follows: new individuals are recruited into the population at a rate of Λ, which increases the number of susceptible individuals. HIV transmission occurs through interactions between susceptible individuals and HIV-infected individuals, such as via anal or vaginal intercourse, sharing of needles and syringes, or other injection equipment, leading to the susceptible individuals becoming HIV-infected. HIV-infected individuals with low-risk behaviours are represented as individuals who have changed their lifestyle, adhere to health protocols as HIV patients, and initiate antiretroviral therapy. The rate of natural mortality is denoted by μ, while the parameter β_1, β_2, and β_3 represent the transmission rates contributed by different classes of infected individuals with $\beta_3 < \beta_2 < \beta_1$. The parameter β_1 denotes the transmission rate from HIV-infected individuals with high-risk behaviours for HIV transmission (I_{hu}), which is typically the highest since they have not changed their behaviour and are not receiving treatment. The parameter β_2 reflects the transmission rate from HIV-infected individuals with low-risk behaviours for HIV transmission (I_{ha}), and is generally lower than β_1 due to behavioural changes and adherence to preventive measures. Meanwhile, the parameter β_3 indicates the transmission rate from HIV-infected individuals on ART therapy

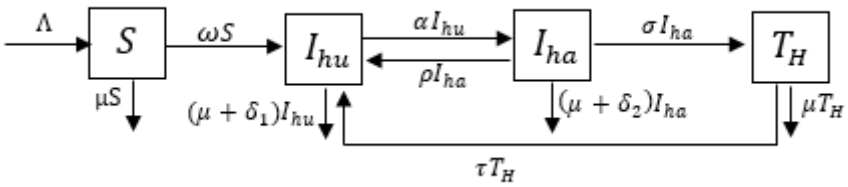

Figure 1. The HIV transmission model compartment diagram.

(T_H), which is usually the smallest because therapy reduces viral load and thus lowers the likelihood of transmission. Together, these parameters form the force of infection ω, which determines the risk of new infections among the susceptible population by accounting for the relative contribution of each infected class. The progression rate from (I_{hu}) to (I_{ha}) is denoted by α. The rate of (I_{ha}) beginning to get bored and abandon habits that cause the disease to spread more easily and enter into (I_{hu}) is described by ρ. The HIV-infected individuals with low-risk behaviours for HIV transmission (I_{ha}) initiate ART therapy at a rate σ and enter into T_H the subpopulation. The rate individuals with HIV and undergoing ART begin to be inconsistent and leave treatment and enter into (I_{hu}) subpopulation is described with τ. The death rate due to HIV disease without treatment is δ_1 and δ_2. HIV infection can ultimately result in death. Without antiretroviral therapy (ART), the immune system of individuals living with HIV gradually deteriorates, progressing to AIDS. At this stage, the susceptibility to opportunistic coinfections increases significantly, which may lead to mortality.

Based on the assumptions, the schematic diagram of the HIV transmission model is as follows: The autonomous system of HIV transmission, as described above and in Figure 1, yields the following equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \omega S - \mu S,\\
\frac{dI_{hu}}{dt} &= \omega S + \rho I_{ha} + \tau T_H - \alpha I_{hu} - \mu I_{hu} - \delta_1 I_{hu},\\
\frac{dI_{ha}}{dt} &= \alpha I_{hu} - \rho I_{ha} - \sigma I_{ha} - \delta_2 I_{ha} - \mu I_{ha},\\
\frac{dT_H}{dt} &= \sigma I_{ha} - \tau T_H - \mu T_H,
\end{aligned} \tag{2}$$

where $\omega = \dfrac{\beta_1 I_{hu} + \beta_2 I_{ha} + \beta_3 T_H}{N}$ and $N(t) = S + I_{hu}(t) + I_{ha}(t) + T_H(t)$. The model (2) subject to the initial conditions by

$$S(0) = S_0 > 0, \quad I_{hu}(0) = I_{hu0} \geq 0, \quad I_{ha}(0) = I_{ha0} \geq 0, \quad \text{and} \quad T_H(0) = T_{H0} \geq 0.$$

All parameters of the model (2) are non-negative because the model (2) describes subpopulations of human interaction. The HIV model (2) has the biologically feasible region on Ω, with

$$\Omega = \left\{ (S, I_{hu}, I_{ha}, T_H) \in \mathbb{R}^4_+ : 0 \leq \mathbb{N} \leq \frac{\Lambda}{\mu} \right\}.$$

The region shown for the model (2) is well-posed and the entire solutions for the initial values belonging to Ω, remains in Ω for every time $t \geq 0$. Therefore, the region Ω is positively invariant. The parameter description that is used on model (2) is written in Table 1.

Table 1. Description of parameters.

Parameters	Description
Λ	Recruitment rate $((individuals)\ (year^{-1}))$
β_1	Transmission rate from $I_{hu}(year^{-1})$
β_2	Transmission rate from $I_{ha}(year^{-1})$
β_3	Transmission rate from $T_H(year^{-1})$
α	The progression rate from I_{hu} to $I_{ha}(year^{-1})$
ρ	The transfer rate from I_{ha} to $I_{hu}(year^{-1})$
σ	The progression rate from I_{ha} to $T_H(year^{-1})$
τ	The transfer rate from T_H to $I_{hu}(year^{-1})$
μ	Natural death rate $(year^{-1})$
δ_1, δ_2	The HIV disease-related death rate $(year^{-1})$

3. Estimation Parameter

In this section, we estimate the parameter values of model (2) using annual HIV case data in Indonesia from 2012 to 2022 [22]. Based on available data, reported annual HIV cases were used as a proxy for the number of individuals in the I_{hu} compartment. The I_{hu} compartment represents HIV-infected individuals who engage in high-risk behaviours and contribute to HIV transmission. Due to data limitations, reported HIV cases were assumed to approximate this compartment for model calibration purposes. We will estimate some initial conditions of the state variables and parameter values utilizing the genetic algorithm method. This method consists of four main stages: forming a population of chromosomes (individuals), selecting parents based on their fitness scores, performing crossover to generate offspring, and introducing random mutations. Parent selection is carried out by choosing two chromosomes from the candidate pool to produce new individuals. Crossover creates one or two new individuals from the selected parent chromosomes, while mutation replaces one or more individuals in the population with new ones. The fitness value for parameter estimation is evaluated using the Mean Absolute Percentage Error (MAPE) [23]. The genetic algorithm was implemented with a population size of 100, a mutation probability of 0.01, and a maximum of 100 generations. The selection rate was set to 0.5 to ensure that the best individuals were retained for the next generation. The estimated values obtained with MAPE= 8.62 can be seen in Table 2.

Table 2. The estimated parameter values

Initial Values/Parameter	Value	Parameters	Value	Parameters	Value
S_0	245,368,689	β_2	0.59	σ	0.0125
I_{ha0}	25,000	β_3	0.08	τ	0.095
T_{H0}	10,000	α	0.012	δ_1	0.58
β_1	0.8101	ρ	0.47	δ_2	0.025

In this study, the actual data is assumed to be I_{hu}. The comparison results of the reported data and the model solution are illustrated in Figure 2. A few things to consider are the total human population of Indonesia in 2012 is 245425200 [24] and the number of HIV cases in Indonesia is 21,511. It is assumed to be the initial value of I_{hu}. The average lifespan in Indonesia at 2012 is 70.2 years [25]. It is assumed, $\mu = \frac{1}{70.2}$ and the parameter Λ can be computed by $\Lambda = \frac{245425200}{70.2}$.

4. Mathematical Analysis

4.1. Mathematical Analysis of the HIV model

Based on model (2), we perform a mathematical analysis to determine the disease-free equilibrium (DFE), the endemic equilibrium (EE), the basic reproduction number R_0, and the stability condition of the DFE. The equilibrium points are obtained by setting the right-hand side of the model

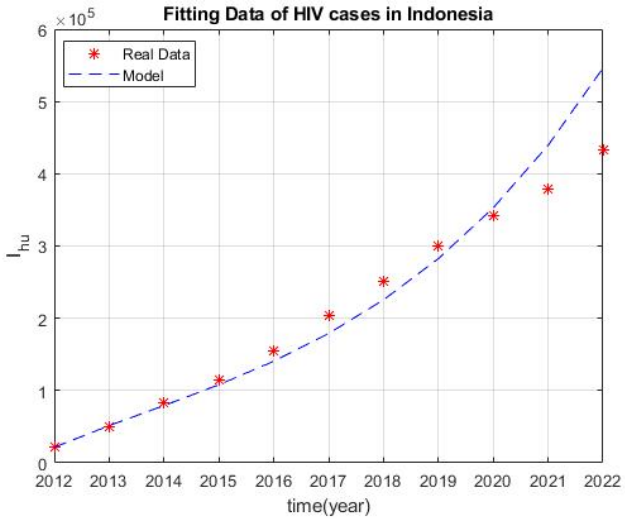

Figure 2. Fitting data of HIV cases in Indonesia.

(2) to zero. The disease-free equilibrium (DFE) points of model (2), denoted by (E_0). The disease-free equilibrium (DFE) of the model (2) is obtained by setting all infected compartments to zero. We get the E_0 of model (2) is $E_0 = (S^0, I_{hu}^0, I_{ha}^0, T_H^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$. The basic reproduction number is the average number of secondary infections generated by a primary infection in a susceptible population, denoted as R_0. The basic reproduction number of model (2) is obtained by using the next-generation matrix method (NGM) [26], a standard method for behavioural dynamics compartmental models. Basic reproduction number R_0 in model (2) is

$$R_0 = \frac{\beta_1 k_2 k_3 + \beta_2 \alpha k_3 + \beta_3 \alpha \sigma}{k_1 k_2 k_3 - \alpha\sigma\tau - \alpha\rho k_3}, \tag{3}$$

with

$$\begin{aligned} k_1 &= (\alpha + \mu + \delta_1), \\ k_2 &= (\rho + \sigma + \mu + \delta_2), \\ k_3 &= (\tau + \mu). \end{aligned}$$

Based on Table 2, the basic reproduction number (R_0) of model (2) is 1.382. The endemic equilibrium (EE) points of model (2) are denoted by E_1. The EE of the system is obtained with assumption $I_{hu} \neq 0$, $I_{ha} \neq 0$ and $T_H \neq 0$. We get the EE of model (2) is $E_1 = (S^*, I_{hu}^*, I_{ha}^*, T_H^*)$, with

$$\begin{aligned} S^* &= \frac{\Lambda}{\omega^* + \mu}, \\ I_{hu}^* &= \frac{\Lambda R_0 - \mu N^*}{R_0(k_1 k_2 k_3 - \alpha\sigma\tau - \alpha\rho k_3)}, \\ I_{ha}^* &= \frac{\alpha}{k_2} I_{hu}^*, \\ T_H^* &= \frac{\alpha\sigma}{k_2 k_3} I_{hu}^*, \end{aligned}$$

where $\omega^* = \dfrac{\beta_1 I_{hu}^* + \beta_2 I_{ha}^* + \beta_3 T_H^*}{N^*}$.

Theorem 1. *The DFE of model (2) is locally asymptotically stable in the feasible region Ω, if $R_0 < 1$, and unstable otherwise.*

Proof. The local stability of the disease-free equilibrium (E_0) is analyzed by linearizing model (2). The Jacobian matrix of the model (2) evaluated at the DFE is denoted $J(E_0)$. The corresponding characteristic equation is obtained from $\det(\lambda I - J(E_0)) = 0$, such that we obtain

$$(\lambda+\mu)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0, \tag{4}$$

with

$$\begin{aligned}
a_1 &= k_1 + k_2 + k_3 - \beta_1, \\
&= \frac{k_1k_2k_3(1-R_0) + k_2^2k_3 + k_2k_3^2 + R_0\alpha(\sigma\tau + \rho k_3) + \alpha(\beta_2k_3 + \beta_3\sigma)}{k_2k_3}, \\
a_2 &= k_1k_2 + k_1k_3 + k_2k_3 - \beta_1(k_2+k_3) - \alpha(\beta_2+\rho), \\
&= k_1k_2 - \alpha\rho + k_3(k_1 + k_2 - \beta_1) + \frac{\beta_3\alpha\sigma}{k_3} - R_0(k_1k_2k_3 - \alpha\sigma\tau - \alpha\rho k_3), \\
a_3 &= (k_1k_2k_3 - \alpha\rho k_3 - \alpha\sigma\tau)(1-R_0).
\end{aligned}$$

From eq. (4) we obtain the first eigenvalue, $\lambda_1 = -\mu$, which is negative. The other eigenvalues, we obtain from the roots of the following equation,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{5}$$

Using the Routh-Hurwitz criteria, the characteristic eq. (5) will have roots with negative real parts if and only if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$. All parameters of the model (2) are positive. The coefficients satisfy $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$ if $R_0 < 1$. Moreover, since $a_1 > 0$ and $a_2 > 0$, their product satisfies $a_1a_2 > a_3$. All the Routh-Hurwitz conditions are satisfied when $R_0 < 1$. Hence, all eigenvalues of the eq. (5) have negative real parts. Therefore, the disease-free equilibrium (E_0) of the model (2) is locally asymptotically stable if $R_0 < 1$. ■

Theorem 2. *If $R_0 < 1$, then the disease-free equilibrium (E_0) of model (2) is globally asymptotically stable in the feasible region Ω.*

Proof. Consider the following Lyapunov function

$$L(\vec{x}) = \varphi_1 I_{hu} + \varphi_2 I_{ha} + \varphi_3 T_H, \tag{6}$$

with $L(\vec{x}) = (S, I_{hu}, I_{ha}, T_H)$.
First, we show that $L(\vec{x}) = 0$ and $L(\vec{x}) > 0$, with $\forall x \neq \vec{x}^* \in W$ and $W \subseteq \mathbb{R}^4_+$. Let $\vec{x}^*$ is DFE of model (2), if $\varphi_1 > 0, \varphi_2 > 0$ and $\varphi_3 > 0$, then $L(\vec{x}) = \varphi_1(0) + \varphi_2(0) + +\varphi_3(0) = 0$ and $L(\vec{x}) > 0$, with $\vec{x} \neq \vec{x}^*$.
Next, we show that $\dfrac{dL}{dt} < 0$,

$$\begin{aligned}
\frac{dL}{dt} &= \varphi_1\frac{dI_{hu}}{dt} + \varphi_2\frac{dI_{ha}}{dt} + \varphi_3\frac{dT_H}{dt}, \\
\frac{dL}{dt} &= \varphi_1(\omega S + \rho I_{ha} + \tau T_H - k_1 I_{hu}) + \varphi_2(\alpha I_{hu} - k_2 I_{ha}) + \varphi_3(\sigma I_{ha} - k_3 T_H).
\end{aligned}$$

Since $\omega = \dfrac{\beta_1 I_{hu} + \beta_2 I_{ha} + \beta_3 T_H}{N}$ and $S \leq N$, we obtain $\omega S \leq \beta_1 I_{hu} + \beta_2 I_{ha} + \beta_3 T_H$. Thus,

$$\frac{dL}{dt} \leq \varphi_1(\beta_1 I_{hu} + \beta_2 I_{ha} + \beta_3 T_H + \rho I_{ha} + \tau T_H - k_1 I_{hu}) + \varphi_2(\alpha I_{hu} - k_2 I_{ha}) + \varphi_3(\sigma I_{ha} - k_3 T_H),$$

$$= (\varphi_1\beta_1 - \varphi_1 k_1 + \varphi_2\alpha) I_{hu} + (\varphi_1\beta_2 + \varphi_1\rho - \varphi_2 k_2 + \varphi_3\sigma) I_{ha} + (\varphi_1\beta_3 + \varphi_1\tau - \varphi_3 k_3) T_H.$$

Let $\varphi_1 = 1$, $\varphi_2 = \dfrac{k_1 - \beta_1}{\alpha}$, and $\varphi_3 = \dfrac{\tau + \beta_3}{k_3}$ we obtain

$$\frac{dL}{dt} = \left(\beta_2 + \rho - \left(\frac{k_1 k_2 - \beta_1 k_2}{\alpha}\right) + \frac{\tau\sigma + \beta_3\sigma}{k_3}\right) I_{ha},$$
$$\frac{dL}{dt} = ((k_1 k_2 k_3 - (\alpha\sigma\tau + \alpha\rho k_3))(R_0 - 1)) I_{ha}.$$

The necessary conditions for $\dfrac{dL}{dt} < 0$ are $R_0 < 1$. Moreover, $\dfrac{dL}{dt} = 0$ only if $I_{ha} = 0$. The largest compact invariant set in $\{(S, I_{hu}, I_{ha}, T_H) \in \Omega,\ \dot{U} = 0\}$ is the singleton $\{E_0\}$. Therefore, by the LaSalle-Lyapunov theorem [27], every solution that starts in Ω approaches E_0 as $t \to \infty$. ■

4.2. Parameter sensitivity analysis

We conduct a sensitivity analysis of parameters to examine how each parameter affects the stability of the equilibrium points, utilizing the sensitivity index (e_ϕ). The parameter values used to compute the sensitivity index are taken from Table 2. The formulation of the sensitivity index is given as follows,

$$e_\phi = \left(\frac{\partial R_0}{\partial \phi}\right) \frac{\phi}{R_0}, \tag{7}$$

The calculation results of the parameter sensitivity index can be seen in Table 3.

Table 3. Sensitivity Index

Parameters	Sensitivity Index Value
β_1	0.983
β_2	0.0165
β_3	0.00025
α	0.0152
ρ	−0.0137
σ	−0.00017
τ	−0.000167
δ_1	−0.975
δ_2	−0.00169

Table 3 shows the sensitivity index's positive or negative values. A positive sensitivity index indicates that an increase in the parameter value leads to a rise in the R_0, while a negative sensitivity index indicates a decline in the R_0 value after an increase in the parameter value. For example, the sensitivity index of β_1 is 0.983, meaning that a 10% rise in β_1 results in approximately a 9.83% rise in the R_0 value. If β_1 value decreases by 10%, then the R_0 value also declines by 9.98%. However, when the ρ value increases by 10%, then the R_0 value decreases to 0.137%. Therefore, ρ has a relatively small effect on changes in the R_0 value. Figure 3 presents simulation illustrating how variations in the values of the β_1 and ρ parameters affect R_0 values and R_{0m} is defined as the basic reproduction number of Model (2) with the parameter values in Table 2.

Figure 4 is a contour plot of the relationship between β_1, ρ, and R_0. Based on Figure 4 show that an increase in β_1 has a significant impact on raising R_0. This indicates that the transmission rate from unaware HIV-infected individuals (I_{hu}) is the dominant factor in determining whether the disease can spread within the population. In contrast, changes in ρ, which represents the transition rate from I_{ha} to I_{hu} compartments, have only a relatively small effect on the dynamics of R_0. The nearly vertical contour lines show that variations in ρ hardly shift the value of R_0 compared to changes in β_1. The line

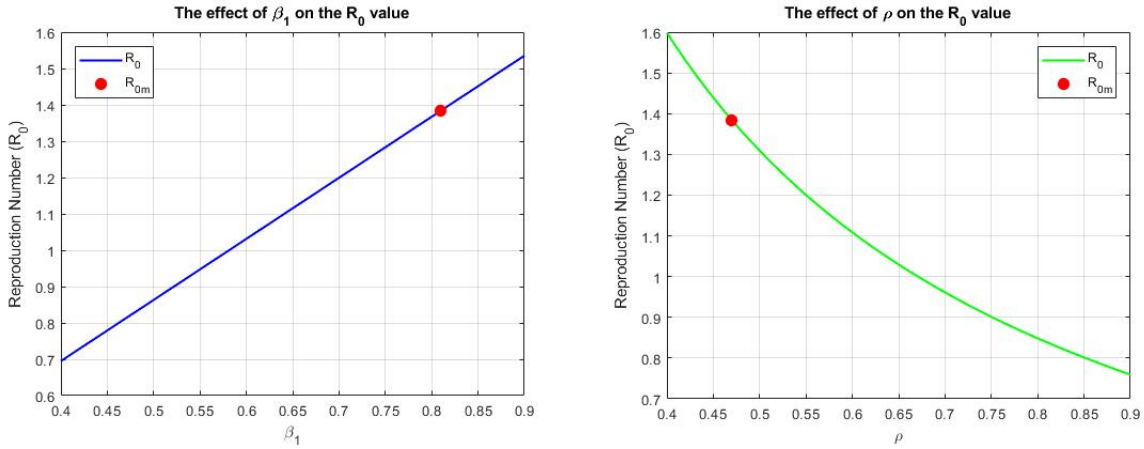

Figure 3. The effect of β_1 and ρ parameters on the R_0 values.

with a value of 1 representing the threshold $R_0 = 1$ separates the disease-free region $(R_0 < 1)$ from the endemic region $(R_0 > 1)$, emphasizing that the main control strategy for reducing HIV transmission should focus on lowering β_1. Thus, interventions that focus solely on ρ are less effective in reducing R_0.

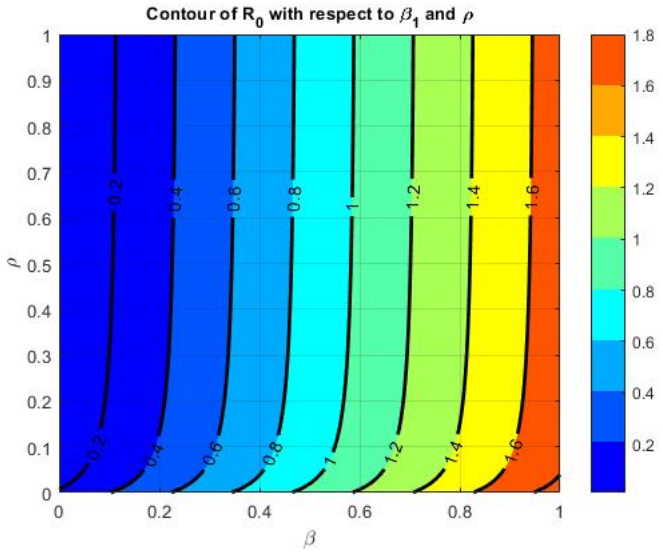

Figure 4. Contour of R_0 with respect to β_1 and ρ.

5. Optimal Control Problem

In this section, the model (2) is extended by introducing time-dependent control variables. This extension is employed to examine the effect of the controls on the model (2). The objective of incorporating controls into model (2) is to make efforts to reduce the number of people infected with HIV and control the spread of the virus throughout the population. Optimal control theory is associated with determining control strategies in dynamic system within certain periods so that the objective function can be optimized. We incorporate two time-dependent controls, i.e., $u_1(t)$ and $u_2(t)$, into model (2). We define $u_1(t)$ as the pre-ART counselling intervention aimed at improving patient readiness for ART, while, $u_2(t)$ represents public health efforts to minimize HIV transmission thru early initiation of ART at the onset of infection.

The extended of HIV transmission model with pre-ART counselling is given by

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \omega S - \mu S,\\
\frac{dI_{hu}}{dt} &= \omega S + \rho I_{ha} + \tau T_H - \theta_1 u_1 I_{hu} - \alpha I_{hu} - \mu I_{hu} - \delta_1 I_{hu},\\
\frac{dI_{ha}}{dt} &= \theta_1 u_1 I_{hu} + \alpha I_{hu} - \rho I_{ha} - \theta_2 u_2 I_{ha} - \sigma I_{ha} - \delta_2 I_{ha} - \mu I_{ha},\\
\frac{dT_H}{dt} &= \theta_2 u_2 I_{ha} + \sigma I_{ha} - \tau T_H - \mu T_H.
\end{aligned} \tag{8}$$

with θ_1 represents success rate of pre-ART counselling and θ_2 represents success treatment rate of HIV-infected individuals. The objective functional or performance index for model (8) defined by:

$$\min J = \int_{t_0}^{t_f} \left(B_1 I_{hu}(t) + B_2 I_{ha}(t) + \frac{C_1}{2} u_1^2 + \frac{C_2}{2} u_2^2 \right) dt, \tag{9}$$

with the constant C_1 and C_2 as weights of relative costs of the controls. The constants B_1 and B_2 are weighting factors that balance the costs associated with the infected compartments I_{hu} and I_{ha}. The main objetive is to find the optimal control values (u_1^* and u_2^*), such that

$$J(u_1^*, u_2^*) = \min \{ J(u_1, u_2) \,|u_1,\ u_2 \in U \}, \tag{10}$$

with $U = \{ (u_1, u_2) \,|0 \le (u_1, u_2) \le 1, \forall t \in [0, t_f] \}$.
Control optimal time span is an interval, $t_0 \le t \le t_f$, which is the stated initial time when control is applied until the end time. Control value input u_1, and u_2 in range $0 \le (u_1, u_2) \le 1$, so applying optimal control gives a success probability of 0 to 1.

The optimal control problem of model (8) is solved by utilizing Pontryagin's maximum principle [28]. The principle converts the minimizing of the cost function eq. (9) with constraint model (8) into the problem of minimizing the Hamiltonian function H. The Hamiltonian function can be defined as follows,

$$H = B_1 I_{hu} + B_2 I_{ha}(t) + + \frac{C_1}{2} u_1^2 + \frac{C_2}{2} u_2^2 + \sum_{i=1}^{5} m_i g_i, \tag{11}$$

with g_i are the state equation or the right-hand side of the model (8) and m_i are costate variables related to state variables. According to Pontryagin's principle, the Hamilton function will achieve an optimal solution when it satisfies both the state equation model (8) and the costate equation, and the condition is stationary. The costate equations are obtained from

$$\dot{m}_1 = -\frac{\partial H}{\partial S},\ \dot{m}_2 = -\frac{\partial H}{\partial I_{hu}},\ \dot{m}_3 = -\frac{\partial H}{\partial I_{ha}},\ \dot{m}_4 = -\frac{\partial H}{\partial T_H},$$

with transversality condition $m_1(t_f) = m_2(t_f) = m_3(t_f) = m_4(t_f) = 0$. The stationary condition H is obtained by minimizing H of control vector $u(t)$ or $\dfrac{\partial H}{\partial u} = 0$. Because the range of $u_1$and u_2 values are $0 \le (u_1, u_2) \le 1$, we get the possibility of $u_1$and u_2 values as follows:

$$u_1^* = \begin{cases} 0 & \text{for} \quad u_1 \le 0 \\ \dfrac{(m_2 - m_3)\,\theta_1 I_{hu}}{C_1} & \text{for} \quad 0 < u_1 < 1 \\ 1 & \text{for} \quad u_1 \ge 1 \end{cases},$$

$$u_2^* = \begin{cases} 0 & \text{for} \quad u_2 \le 0 \\ \dfrac{(m_3 - m_4)\,\theta_2 I_{ha}}{C_2} & \text{for} \quad 0 < u_2 < 1 \\ 1 & \text{for} \quad u_2 \ge 1 \end{cases}.$$

Based on the possibilities above, the optimal control values are obtained as follows:

$$\begin{aligned} u_1^* &= \max\left(0, \min\left(\frac{(m_2 - m_3)\,\theta_1 I_{hu}}{C_1}, 1\right)\right), \\ u_2^* &= \max\left(0, \min\left(\frac{(m_3 - m_4)\,\theta_2 I_{ha}}{C_2}, 1\right)\right). \end{aligned} \tag{12}$$

The optimal system is acquired by substituting the optimal control variables u_1^* and u_2^* into the system of state equations and costate equations. The method used to solve the optimal control problem is a backward and forward sweep using the fourth-order Runge-Kutta method.

6. Numerical results

In this part, we will show some simulations of optimal control of the HIV transmission model with pre-ART counselling and treatment model (8). The initial conditions and parameter values used in this simulation are based on Table 2 with $B_1 = 1, B_2 = 1, C_1 = 2, C_2 = 5, \theta_1 = 0.8, \theta_2 = 0.9$ and the time interval (t) used in the simulation is $t \in [0,5]$. The unit of time used is years. The results of the model simulation with optimal control are presented as follows:

a. Scenario I: implementation of control strategies with u_1 control only.

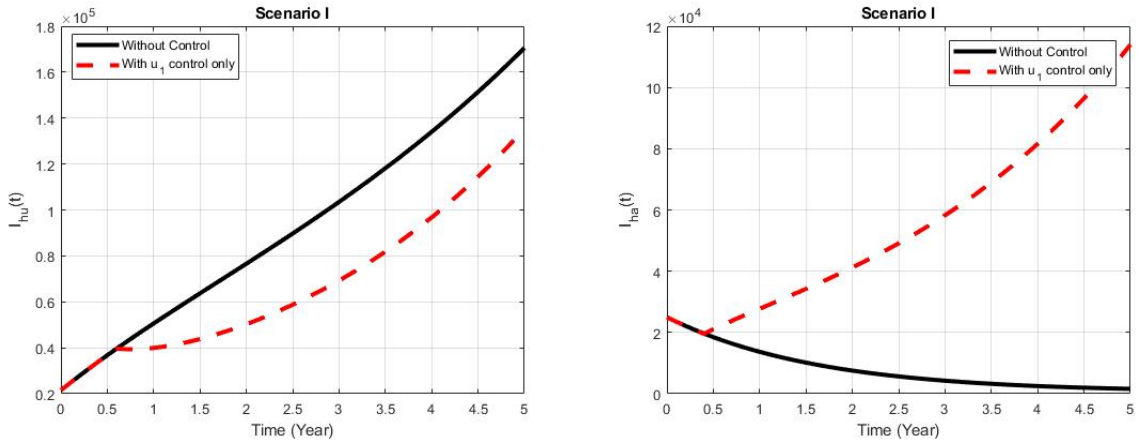

Figure 5. The effect of scenario I on the number of I_{hu} and I_{ha} within 5 years.

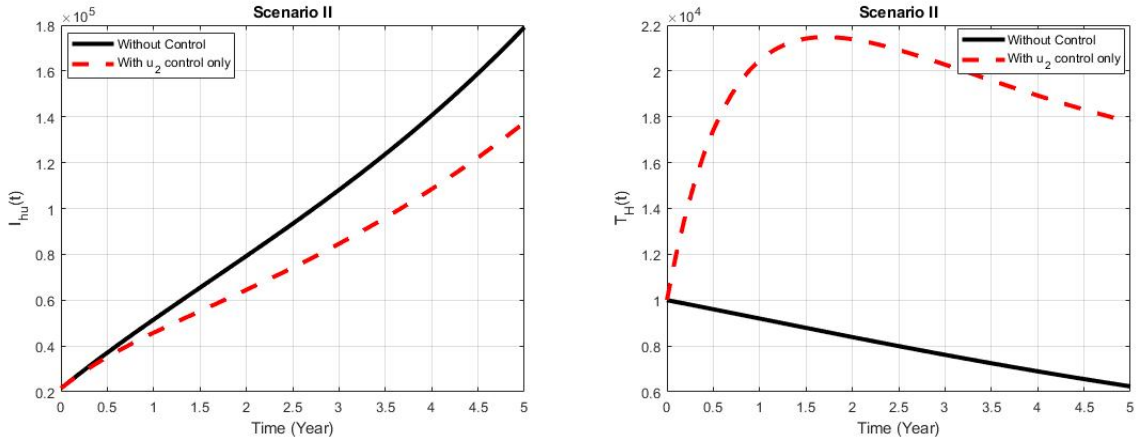

Figure 6. The effect of scenario II on the number of I_{hu} and T_H within 5 years.

Based on Figure 5, implementation of u_1 control only over a five-year period shows a limited on reducing the number of HIV-infected individuals with high-risk behaviours (I_{hu}). Although the

number of individuals in the I_{hu} compartment still increases over time, the growth rate is slower compared to the uncontrolled case. In addition, the implementation of u_1 increases the number of HIV-infected individuals with low-risk behaviours (I_{ha}), indicating that pre-ART counselling encouraged HIV-infected individuals to adopt healthier lifestyles, adhere to health protocols, and voluntarily initiate treatment. When the scenario I is implemented, the number of I_{ha} shows an increase relative to the condition without control measures. At time $t = 0$, the number of HIV-infected individuals with high-risk behaviours was 21,551. Based on model simulations without controls, this number increased to 170,530 individuals at $t = 5$. In the same period, the number of I_{ha} individuals declined from 25,000 to 1,589. Implementation of strategy I is quite effective in decreasing the number of I_{hu} and increasing the motivation of I_{ha} to undergo ART treatment, as shown by the dotted red curve, which exhibits an upward trend and reaches $113,960$ individuals at $t = 5$.

b. Scenario II: implementation of control strategies with u_2 control only.
Based on Figure 6, In the first figure, the number of HIV-infected individuals with high-risk behaviour I_{hu} increases steadily over time in both scenarios. However, when control u_2 is applied, the growth of I_{hu} becomes slower compared to the uncontrolled case. This indicates that the treatment intervention contributes to reducing the progression of individuals who maintain high-risk behaviours, although the number of infected individuals still increases over time. In the second figure, the number of individuals receiving ART T_H shows a significantly different pattern between the uncontrolled and controlled scenarios. Without control measures, the number of individuals undergoing treatment gradually decreases over time. In contrast, when the treatment control $u_2$is implemented, the number of individuals receiving ART increases rapidly during the early stage of the intervention and reaches a peak around the second year before gradually declining.

c. Scenario III: implementation of control strategies with the combination u_1 and u_2.

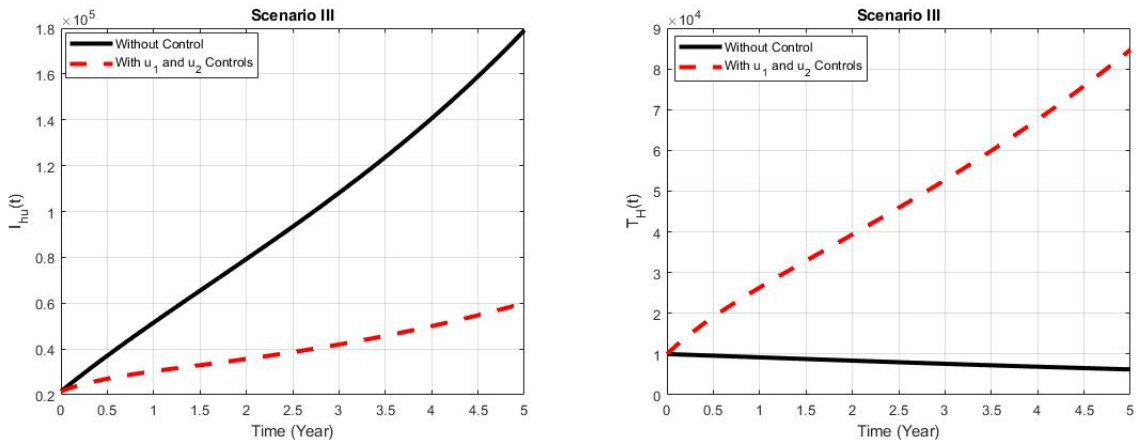

Figure 7. The effect of scenario III on the number of I_{hu} and T_H within 5 years.

Based on Figure 7, implementation of combined control u_1 and u_2 over a 5-year period can significantly reduce the spread of HIV disease. Under scenario III, there is a substantial reduction in the number of HIV-infected individuals with high-risk behaviours (I_{hu}) and a significant increase in the number of HIV-infected individuals on ART T_H compared to the no-control case. At time $t = 5$, the number of HIV-infected individuals with high-risk behaviours for HIV transmission without control is 179,050, whereas under scenario III it is reduced to 60,019 individuals. Meanwhile, the number of HIV-infected individuals on ART without control is 6,237, and with scenario III it reaches 84,679 individuals.
Figure 8 illustrates the impact of Scenario III on the cumulative number of HIV infections over

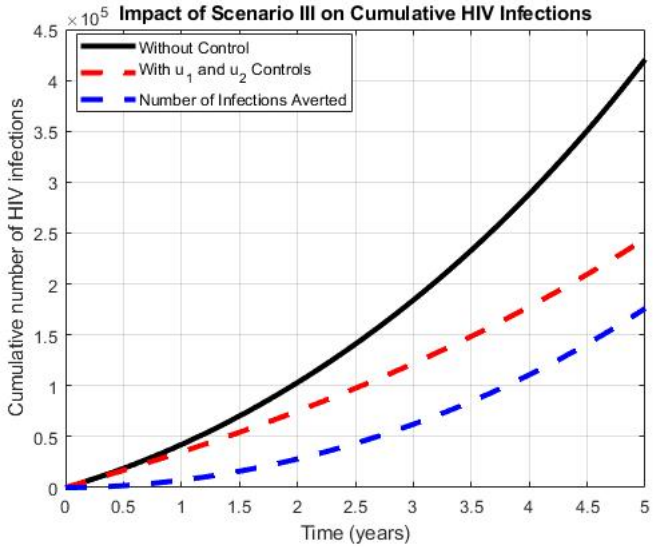

Figure 8. The impact of scenario III control strategies on cumulative HIV infections.

a five-year simulation period. The black solid curve represents the cumulative number of HIV infections in the absence of control measures, while the red dashed curve represents the cumulative infections under the combined control strategiesu_1 and u_2. The blue dashed curve shows the number of infections averted due to the implementation of the control strategies. As observed in the figure, the cumulative number of HIV infections without control increases rapidly throughout the simulation period, indicating the continuous spread of HIV in the population when no intervention strategies are applied. In contrast, when the combined control strategies are implemented, the cumulative number of infections grows at a much slower rate. The increasing gap between the uncontrolled and controlled trajectories demonstrates the effectiveness of the intervention strategies in reducing HIV transmission. The blue curve indicates that the number of infections averted increases steadily over time, suggesting that the benefits of the control strategies accumulate as the intervention period progresses. By the end of the five-year simulation period, the combined control strategies prevent approximately 42% of potential HIV infections compared to the uncontrolled scenario. This result confirms that the simultaneous implementation of behavioural interventions and treatment strategies plays a crucial role in significantly reducing HIV transmission in the population.

The simulation of the control profiles at time t can be seen in Figure 9. Based on Figure 9, the application of the combined control strategy $u_1$and u_2 can significantly reduce the number of HIV-infected individuals with high-risk behaviours for HIV transmission(I_{hu}) and increase the number of HIV-infected individuals on ART (T_H). In scenario III, the implementation of u_1 which represents pre-ART counselling, provides maximum impact from the beginning until $(t = 4.8)$, after which it decreases significantly until the end of the period. The implementation of u_2, which represents ART treatment for HIV-infected individuals, provides maximum impact from the beginning until $t = 4.7$ and decreases gradually until the end of the period. Control is terminated at the end of the period, which means no further intervention is provided. Three scenarios have been implemented and simulated. The best strategy is the third scenario, which shows that the optimal application of control strategies in the model is highly effective in controlling the spread of HIV in the population.

7. Conclusion

In this paper, we studied a deterministic model of HIV transmission with the existence of pre-antiretroviral therapy counselling and treatment consisting of four compartments. Mathematical anal-

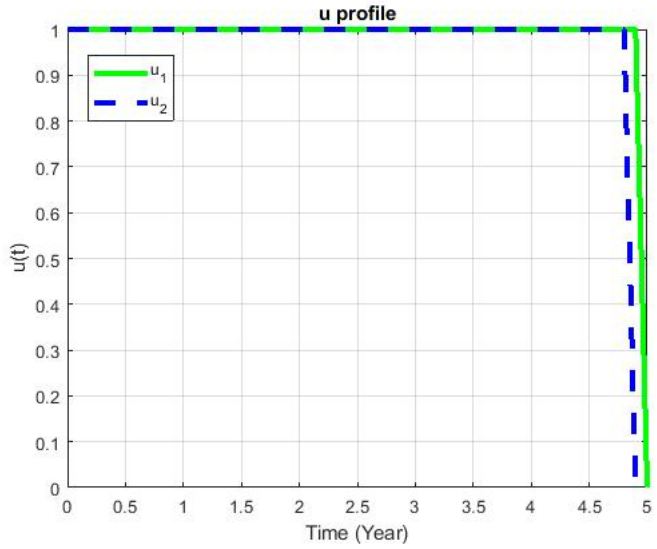

Figure 9. The control profiles in Scenario III.

ysis of the model obtained disease-free and endemic equilibrium points, basic reproduction numbers, and stability analysis around the disease-free equilibrium point. The stability criteria of disease-free equilibrium points are globally asymptotically stable when the basic reproduction number is less than one. The mathematical model of HIV transmission was developed by incorporating control variables. There are two time-dependent control variables in the model, i.e., pre-antiretroviral therapy counselling (u_1) and antiretroviral therapy (u_2) for individuals with HIV. The aim of providing optimal control is to investigate the impact of control strategies on the transmission of HIV disease and minimize the population of unaware HIV-infected individuals. Pontryagin's minimum principle method is used to solve the necessary conditions for optimal control strategies. Based on the simulation results, the implementation of scenario III is the most effective control strategy for reducing HIV transmission in the population, and the number of infections averted increases steadily over time. Future research could further enhance this model by incorporating co-infection dynamics, fractional-order derivatives, and additional control variables to capture of HIV transmission patterns. Integrating region-specific HIV data from across Indonesia would improve model calibration and strengthen its applicability to public health policy. The findings underscore the importance of reinforcing pre-antiretroviral therapy counselling and promoting early ART initiation through sustained governmental commitment and community-based interventions as integral components of national strategies for HIV control and elimination in Indonesia.

Supplementary Information

Author Contributions. **Mohamad Syafi'i:** Writing-original draft preparation, conceptualization, methodology, formal analysis, software. **Fatmawati:** Writing-review & editing, supervision, conceptualization, methodology, validation. **Ahmadin:** Writing-review & editing, supervision, validation. **Chidozie Williams Chukwu:** Writing-review & editing, supervision, validation.

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Data availability. The data used in the study were obtained from publicly available sources or previously

published research. All data sources have been appropriately cited in the references section.

Abbreviations.

HIV	:	Human Immunodeficiency Virus
AIDS	:	Acquired Immunodeficiency Syndrome
ART	:	Antiretroviral Therapy
DFE	:	Disease-free Equilibrium
EE	:	Endemic Equilibrium

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