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Anaysis of Predator-Prey Dynamics Using Holling Type I & II Response Functions with Kleptoparasitism and Anti-Predator Behavior

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Abstract. *Predator-prey interactions involving 3 species in an African forest ecosystem between Deer, Hyena and Lion considering the influence of kleptoparasitism and anti-predator behaviour using Holling type I & II functional responses. This predator prey model is constructed based on the assumption that the behaviour of the second predator Hyena often has the ability to defend itself against other predators such as fleeing, fighting, and intimidating which is called anti-predator behaviour. Based on the existing phenonema, the objectives of this study are to determine the model construction, equilibrium point analysis and stability, as well as numerical simulation and interpretation of the prey-prey model using Holling type I & II functional responses in the presence of kleptoparasitism and anti-predator behaviour. The calculation analysis in this study was carried out by finding the equilibrium point and stability analysis. The results of the dynamic analysis show that there are five equilibrium points with the type of stability, namely* $E_1(x,y,z) = (0,0,0)$ *which states the extinction of the three populations, point equilibrium* $E_2(x,y,z) = (K,0,0)$ *which represents the extinction of the first predator and second predator populations, point equilibrium* $E_3(x,y,z) = \left(-\frac{\theta_2}{\theta_2 b-\mu_2}, 0, -\frac{r\mu_2(K(\beta\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu)^2}\right)$ *which expresses extinction in the first predator population, point equilibrium* $E_4(x,y,z) = \left(\frac{\theta_1}{\mu_1}, \frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}, 0\right)$ *which expresses extinction at the second predator, and equilibrium point* $E_5(x^*,y^*,z^*)$ *which states that all three populations can coexist. Numerical simulation results show the existence of double stability at points* E_4 *and* E_6 *when the parameter values* $\mu_1 = 0.3, \mu_2 = 0.12$ *and double stability occurs again at points* E_4 *and* E_3 *when the parameter variation values* $\mu_1 = 0.3, \mu_2 = 0.158$.

1. Introduction

Ecology is a branch of biology that studies the reciprocal relationships between living things and their environment [1]. In an ecosystem, interactions between populations are very close and influence each other, where every living thing always depends on other living things for its survival. The relationships that occur between populations of living things in an ecosystem have several characteristics, including being beneficial, competitive, predatory, or harmful to one party, such as parasitism [2].

Ecological interactions in which one parasitic organism benefits at the expense of another organism, known as the host, are called parasitism [3]. Parasites usually depend on their hosts for food, shelter, or other resources necessary for survival. This relationship is often unbalanced, with the host suffering losses such as the loss of nutrients, energy, or even death. Parasitism can be classified into several types based on the degree of dependence of the parasite on the host [4]. One form of parasitism in ecology is kleptoparasitism. Kleptoparasitism is a strategy of obtaining food by stealing or taking it from other species that have already obtained it. The term "kleptoparasitism" describes the act of individuals of one species taking food from individuals of another species [5]. This behavior is also known as 'piracy', 'seizure', 'food parasitism' or 'theft' because it involves taking resources, especially food, from other individuals who have hunted it first [6–8].

Kleptoparasitism is classified into three main forms based on its interaction strategy: open aggression, in which kleptoparasites use direct violence or threats of violence to take control of food belonging to other predators; competitive seizure, in which passive individuals take advantage of resources found by other more active individuals without resorting to direct violence; but still involving competition for food, and covert kleptoparasitism, which is the act of taking or stealing food without getting involved in a fight [9]. Kleptoparasitism allows species to save time and energy that would otherwise be spent searching for food, as they obtain food by stealing it from other species. This allows them to focus their time and energy on other activities that are important for survival and reproduction, such as finding a mate [10]. In reality, many species in nature exhibit kleptoparasitism, including hyenas [11], crows [12], insects [13], spiders [14], etc.

The main factor in the predator-prey relationship is the way predators prey on their prey, known as functional response. Holling introduced three types of functional responses, namely Holling type I, type II, and type III functional responses, which describe prey density responses in different ways [15]. Subsequently, Monod and Haldane conducted research related to functional response, which became known as Holling type IV functional response [16]. This study considers Holling type I and type II functional responses as an approach to modeling interactions between two predators, namely lions as the first predator and hyenas *(Crocuta crocuta)* as the second predator, which are kleptoparasitic, with deer as the main prey for both because deer have an abundant population. Holling type I is used to describe predators that engage in kleptoparasitism, i.e., obtaining food by stealing from other predators without considering prey handling time. Meanwhile, Holling type II is used to describe predators that actively hunt and experience prey handling time.

Every prey that interacts with a predator is assumed to be caught and eaten by the predator [17]. However, in reality, not all prey that encounter a predator are successfully captured and consumed by the predator. Some prey have defense mechanisms, such as the ability to flee or even fight back against predators. In addition, some prey species employ mobbing strategies, which is behavior where a group of individuals work together to drive away or intimidate predators. The behavior of prey that is able to avoid and fight predators is called anti-predator behavior [18]. This phenomenon of anti-predator behavior among predators can be found in the interactions between lions *(Panthera leo)* and hyenas *(Crocuta crocuta)* living in African ecosystems. Although hyenas are predators, they are often the target of lions' aggression in obtaining food. In response to this threat, hyenas have adapted various mechanisms to avoid lion attacks, such as mobbing behavior, where hyenas aggressively try to chase lions away [19]. In certain situations, especially when hyenas greatly outnumber lions, hyenas not only survive but can also attack, particularly young, injured, or solitary lions [20]. In the Ngorongoro Crater, lion deaths are entirely attributable to hyena aggression, particularly when no adult male lions are present [21]. As a form of protection, lions also exhibit anti-predator behavior to protect themselves and their group. Lions form social groups called prides, in which members work together to guard their territory, protect lion cubs, and face threats from hyenas [22]. In the same ecosystem, deer are one of the most abundant herbivores and live in groups, making them a prime target for large predators such as lions and hyenas because they are easy to detect and hunt. Based on this background, the author is interested in mathematically examining the predator-prey interaction in African forest ecosystems between deer as prey, lions as primary predators, and hyenas as secondary predators.

Based on previous predator-prey model studies, the researcher is interested in reconstructing the predator-prey model developed by previous researchers, taking into account anti-predator behavior without competition between predators. This study considers the same behavior as previous researchers, namely the existence of kleptoparasitism. Unlike the model developed by [10], which does not consider anti-predator behavior, this study uses a different approach, namely by using the aspect of competition in interactions between predators. In addition, this study also refers to the predator-prey model developed by [23], which discusses a two-population system and incorporates anti-predator behavior in the prey population. Furthermore, the study of anti-predator behavior is also discussed by [24], who developed a model of two competing prey species with one predator, in which one prey species exhibits anti-predator behavior. Considering these aspects of kleptoparasitism and anti-predator behavior, this study is titled "Analysis of predator-prey dynamics using Holling type I & II respinse functions with kleptoparasitism and anti-predator behaviour ".

2. Model Formulation

Kleptoparasitism is a strategy of obtaining food by taking it from other species [10]. discussing the predator-prey model with kleptoparasitism, where the prey population is assumed to grow logistically. In this model, the predator population does not only depend on the number of prey, but is also influenced by interactions with kleptoparasitic species represented by the parameter α. The greater the value of α, the stronger the impact of kleptoparasitism in reducing the effectiveness of predators in capturing prey. The ability of predators to capture prey is influenced by kleptoparasitism, which forms a relationship between prey populations, predators, and kleptoparasites. The following is the equation with the presence of kleptoparasitic species.

$$F(\alpha,z)=\frac{1}{1+\alpha z},$$

where α represents the degree of influence of kleptoparasitism. Here is the Predator-Prey model, with one prey and two predators, where one predator is kleptoparasitic [10].

$$\begin{aligned}
\frac{dx}{dt}&=rx\left(1-\frac{x}{K}\right)-\beta_1xy-\beta_2xz,\\
\frac{dy}{dt}&=\frac{\mu_1xy}{1+\alpha z}-k_1yz-\theta_1y,\\
\frac{dz}{dt}&=\mu_2xz-k_2yz-\theta_2y+\mu_3xyz.
\end{aligned}\tag{1}$$

All parameters are positive values, where x is the prey population, y is the predator population, z is the kleptoparasitic predator population, r is the prey growth rate, and K is the carrying capacity of the environment for prey. Parameters β_1 and β_2 represent the predation rates by predators and kleptoparasitic species, respectively. In addition, μ_1, μ_2, and μ_3 represent the conversion of prey energy into predator energy and kleptoparasitism species, as well as direct predation and theft of prey catches by kleptoparasitism species. Parameters k_1 and k_2 indicate the level of competition for predators and kleptoparasitism species. Furthermore, θ_1 and θ_2 represent the natural mortality rate of predators, and α represents the level of influence of kleptoparasitism. This study models the interaction between three species, considering the Holling type II functional response for the second predator as an active predator, and Holling type I when the second predator is a kleptoparasite without prey handling time, by combining the influence of anti-predator behavior between predators denoted by the parameter η

to obtain the following new model construction.

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{\beta_1 xy}{1+bx}-\frac{\beta_2 xz}{1+bx},\\ \frac{dy}{dt} &= \frac{\mu_1 xy}{1+\alpha z}-\theta_1 y,\\ \frac{dz}{dt} &= \frac{\mu_2 xz}{1+bx}-\theta_2 z+\mu_3 xyz-\eta yz.\end{aligned} \tag{2}$$

All parameters are positive values, where x is the prey population, y is the predator population, z is the kleptoparasitic predator population.

3. Result and Discussion

3.1. Equilibrium Point Analysis

The equilibrium point in eq. (2) is obtained by solving $\frac{dx}{dt}=0, \frac{dy}{dt}=0, \frac{dz}{dt}=0$. The equilibrium points of the system are obtained as follows.

1. Equilibrium point $E_1(x,y,z)=(0,0,0)$
   The equilibrium point E_1 states that the prey, first predator, and second predator populations are extinct.
2. Equilibrium point $E_2(x,y,z)=(K,0,0)$
   The equilibrium point E_2 states that the prey population does not experience extinction, while the first predator and second predator populations experience extinction.
3. Equilibrium point $E_3(x,y,z)=\left(-\frac{\theta_2}{\theta_2 b-\mu_2},0,-\frac{e\mu_2(K(\beta\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu)^2}\right)$.
   The equilibrium point E_3 states that the prey population and the second predator do not experience extinction, while the first predator population experiences extinction. Exists when $x=\left(-\frac{\theta_2}{\theta_2 b-v_2}\right)$ is positive with the condition $\theta_2 b-\mu_2<0$ or equivalent to $\mu_2>\theta_2 b$ and exist when $z=\left(-\frac{r\mu_2(K(b\theta_2-\mu_2)+\theta_2}{K\beta_2(b\theta_2-\mu)^2}\right)$ is positive with the condition $\mu_2>\theta_2(b+\frac{1}{K})$. by taking the intersection of the two conditions Equilibrium point E_3. Then equilibrium point E_3 exists if $\mu_2>\theta_2(b+\frac{1}{K})$ which is obtained from the intersection of the x and y point conditions.
4. Equilibrium point $E_4(x,y,z)=\left(\frac{\theta_1}{\mu_1},\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1},0\right)$
   The equilibrium point E_4 states that the prey population and the first predator do not experience extinction or exist, while the second predator population experiences extinction. The equilibrium point $E4$ exists when $\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}>0$ The result is positive with the condition or equivalent to $\mu_1>\frac{\theta_1}{K}$.
5. Equilibrium point $E_5(x^*,y^*,z^*)$
   Equilibrium point E_5 states that the prey, first predator, and second predator populations do not experience extinction or exist. is obtained when all populations persist. The equilibrium values are derived as:

$$\begin{aligned}y^* &= \frac{(\theta_2(1+bx^*)-\mu_2 x^*)}{(1+bx^*)(\mu_3 x^*-\eta)},\\ z^* &= \frac{(rK\mu_3 x^*-rk\eta+rbK\mu_3 x^{*2}-rbK\eta x^*-r\mu_3 x^2+r\eta x^*-rbx^{*3}+rb\eta x^{*2}-\beta_1\theta_2 K-\beta_1\theta_2 Kbx^*+\beta_1 K\mu_2 x^*)}{\beta_2 K\mu_3 x^*-\beta_2 K\eta},\\ 0 &= a_1 x^{*3}+a_2 x^{*2}+a_3 x^*+a_4.\end{aligned}$$

With each of the above components as follows.

$$\begin{aligned}a_1 &= \theta_1\alpha rb\mu_3,\\ a_2 &= \mu_1\beta_2 K\mu_3-\theta_1\alpha rKb\mu_3+\theta_1\alpha r\mu_3+\theta_1\alpha rb\eta,\end{aligned}$$

$$a_3 = -\mu_1\beta_2K\eta - \theta_1\beta_2K\mu_3 - \theta_1\alpha rK\mu_3 + \theta_1\alpha rKb\eta - \theta_1\alpha r\eta + \theta_1K\beta_1\theta_2 b - \theta_1\alpha K\beta_1\mu_2,$$
$$a_4 = \theta_1\alpha rK\eta + \theta_1\beta_2K\eta + \theta_1\alpha K\beta_1\theta_2.$$

The above equation is a third-degree nonlinear polynomial. This means that this equation can produce up to three solution values for x^*, depending on the parameter values used. Each value of x^* is substituted into the equations y^* and z^*, resulting in three different combinations of equilibrium points. These three equilibrium points, as long as they meet the biological condition (positive value), are each denoted as E_5, E_6, E_7.

3.2. Local Stability

Stability analysis can be determined by linearizing the system arranged in matrix form. The Jacobi matrix is obtained by finding the derivative of each eq. (2), which are denoted as follows.

$$J(x,y,z) = \begin{bmatrix} r(1-\frac{x}{K}) - \frac{x}{K} - \beta_1 y - \frac{\beta_2 z}{1+bx} + \frac{\beta_2 xzb}{(1+bx)^2} & -\beta_1 x & -\frac{\beta_2 x}{1+bx} \\ \frac{\mu_1 y}{1+\alpha z} & \frac{\mu_1 x}{1+\alpha z} - \theta_1 & -\frac{\mu_1 xy\alpha}{(1+\alpha z)^2} \\ \frac{\mu_2 z}{1+bx} - \frac{\mu_2 xzb}{(1+bx)^2} + \mu_2 yz & \mu_3 xz - \eta z & \frac{\mu_2 x}{1+bx} - \theta_2 + \mu_3 xy - \eta y \end{bmatrix} \tag{3}$$

Theorem 1. *The equilibrium point $E_1(0,0,0)$ is always unstable. Point E_1 is always unstable, with the type of instability being a saddle point.*

Proof. Substituting $E_1(x,y,z) = (0,0,0)$ into the Jacobi matrix in eq. (3) yields.

$$J_{E_1} = \begin{bmatrix} r & 0 & 0 \\ 0 & -\theta_1 & 0 \\ 0 & 0 & -\theta_2 \end{bmatrix} \tag{4}$$

All parameters are assumed to be positive. It is known that $\lambda_1 > 0$, so $\lambda_1 > 0$, -$\theta_1 < 0$, so $\lambda_2 < 0$, and -$\theta_2 < 0$, so $\lambda_3 < 0$. Because $\lambda_1 > 0, \lambda_2 < 0$, and $\lambda_3 < 0$ the equilibrium point $E_1(x,y,z) = (0,0,0)$ is unstable with a saddle point instability. ■

Theorem 2. *The equilibrium point stability $E_2(x,y,z) = (K,0,0)$. Asymptotically stable Node when $\mu_1 < \frac{\theta_1}{K}$ and $\mu_2 < \theta_2(b+\frac{1}{K})$.*

Proof. Substituting $E_2(x,y,z) = (K,0,0)$ into the Jacobi matrix in eq. (3) yields.

$$J_{E_2} = \begin{bmatrix} -r & -\beta_1 K & -\frac{\beta_2 K}{1+bK} \\ 0 & \mu_1 K - \theta_1 & 0 \\ 0 & 0 & \frac{\mu_2 K}{1+bK} - \theta_2 \end{bmatrix}$$

All parameters are assumed to be positive. Thus, the following eigenvalues are obtained $\lambda_1 = -r, \lambda_2 = \mu_1 K - \theta_1$, dan $\lambda_3 = -\frac{(\theta_2 Kb + \theta_2 - \mu_2 K)}{1+bx}$. Asymptotic stability of the node when $\lambda_1 < 0, \lambda_2 < 0$, and $\lambda_3 < 0$. It is known that $-r < 0$, so $\lambda_1 < 0$. conditions that must be met when $\lambda_2 < 0$ and $\lambda_3 < 0$ with conditions $\mu_1 K - \theta_1 < 0$ or $\mu_1 < \frac{\theta_1}{K}$, and $-\frac{(\theta_2 Kb + \theta_2 - \mu_2 K)}{1+bK}$ with values $1 + bK > 0$. ■

Theorem 3. *The equilibrium point stability $E_3(x,y,z) = \left(-\frac{\theta_2}{\theta_2 b - \mu_2}, 0, \frac{-r\mu_2(K(b\theta_2 - \mu_2) + \theta_2)}{K\beta_2(b\theta_2 - \mu_2)^2}\right)$ asymptotically stable if $A_22 < 0, T_1 < 0$, and $D1 > 0$.*

Proof. Substituting $E_3(x,y,z) = \left(-\frac{\theta_2}{\theta_2 b - \mu_2}, 0, \frac{-r\mu_2(K(b\theta_2 - \mu_2) + \theta_2)}{K\beta_2(b\theta_2 - \mu_2)^2}\right)$ into the Jacobian matrix in eq. (3) yields.

$$J_{E_3} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ 0 & A_{22} & 0 \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

with each of its components being,

$$A_{11} = r\left(1-\frac{\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}{K}\right)-\frac{r\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}{K}-\frac{\beta_2\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu_2)^2}\right)}{1+b\left(-\frac{\theta_2}{\theta_2 b\mu_2}\right)}+\frac{\beta_2\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu_2)^2}\right)b}{\left(1+b\left(-\frac{\theta_2}{\theta_2-\mu_2}\right)\right)^2},$$

$$A_{21} = -\beta_1\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right),$$

$$A_{31} = -\frac{\beta_2\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}{1+b\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)},$$

$$A_{22} = \frac{\mu_1\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}{1+\alpha\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu)^2}\right)},$$

$$A_{13} = \frac{\mu_2\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu_2)^2}\right)}{1+b\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}-\frac{\mu_2\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2\mu_2)}\right)b}{\left(1+b\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)\right)^2},$$

$$A_{23} = \mu_3\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu_2)^2}\right)-\eta\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu_2)^2}\right),$$

$$A_{33} = \frac{\mu_2\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}{1+b\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}-\theta_2.$$

The eigenvalues are found by the characteristic equation $\det(J_{E_3}-\lambda I)=0$.

$$(A_{22}-\lambda)[\lambda^2-\lambda(A_{11}+A_{33})+A_{11}A_{33}-A_{31}A_{13}]=0.$$

Obtained eigenvalue $\lambda_1 = A_{22}$, for λ_2 and λ_3 obtained from $\lambda^2-\lambda T_1+D_1=0$. Where $T_1 = A_{11}+A_{33}$ and $D_1 = A_{11}A_{33}-A_{31}A_{13}$. The equilibrium point E_3 is asymptotically stable if $A_{22}<0, T_1<0$ and $D_1>0$. Thus, the obtained eigenvalues are: $\lambda_1 = \frac{\mu_1\left(-\frac{\theta_2}{\theta_2 b-\mu_2}\right)}{1+\alpha\left(\frac{-r\mu_2(K(b\theta_2-\mu_2)+\theta_2)}{K\beta_2(b\theta_2-\mu)^2}\right)}$, $\lambda_2,\lambda_3 = \frac{T_1\pm\sqrt{T_1^2-4aD_1}}{2a} = \frac{(A_{11}A_{33})\pm\sqrt{(A_{11}A_{33})^2-4(A_{11}A_{33}-A_{31}A_{13})}}{2}$. ■

Theorem 4. *The equilibrium point stability* $E_4(x,y,z)=\left(\frac{\theta_1}{\mu_1},\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1},0\right)$ *asymptotically stable if* $B_33<0, T_2<0$, *and* $D_2>0$.

Proof. Substituting $E_4(x,y,z)=\left(\frac{\theta_1}{\mu_1},\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1},0\right)$ into the Jacobian matrix in eq. (3) yields.

$$J_{E_4} = \begin{bmatrix} B_{11} & B_{21} & B_{31}\\ B_{12} & B_{22} & B_{32}\\ 0 & 0 & B_{33}\end{bmatrix},$$

with each of its components being,

$$B_{11} = r\left(1-\frac{\left(\frac{\theta_1}{\mu_1}\right)}{K}\right)-\frac{r\left(\frac{\theta_1}{\mu_1}\right)}{K}-\beta_1\left(\frac{r(\mu_1-\theta_1)}{\mu_1 K\beta_1}\right),$$

$$B_{21} = -\beta_1\left(\frac{\theta_1}{\mu_1}\right),$$

$$B_{31} = -\frac{\beta_2\left(\frac{\theta_1}{\mu_1}\right)}{1+b\left(\frac{\theta_1}{\mu_1}\right)},$$
$$B_{12} = \mu_1\left(\frac{r(\mu_1-\theta_1)}{\mu_1 K\beta_1}\right),$$
$$B_{22} = \mu_1\left(\frac{\theta_1}{\mu_1}\right)-\theta_1,$$
$$B_{32} = -\mu_1\left(\frac{\theta_1}{\mu_1}\right)\left(\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}\right),$$
$$B_{33} = \frac{\mu_2\left(\frac{\theta_1}{\mu_1}\right)}{1+b\left(\frac{\theta_1}{\mu_1}\right)}-\theta_1+\mu_2\left(\frac{\theta_1}{\mu_1}\right)\left(\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}\right)-\eta\left(\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}\right).$$

The eigenvalues are found by the characteristic equation $\det(J_{E_4}-\lambda I=0)$, as shown below.

$$(B_{33}-\lambda)[\lambda^2-\lambda(B_{11}+B_{22})+B_{11}B_{22}-B_{21}B_{12}]=0.$$

Obtained eigenvalue $\lambda_1 = B_{33}$, for λ_2 and λ_3 obtained from $\lambda^2-\lambda T_2+D_2=0$. Where $T_2 = B_{11}+B_{22}$ and $D_2 = B_{11}B_{22}-B_{12}B_{21}$. The equilibrium point E_4 is asymptotically stable if $B_{33}<0, T_2<0$ and $D_2>0$. Thus, the obtained eigenvalues are: $\lambda_1 = B_{33} = \frac{\mu_2\left(\frac{\theta_1}{\mu_1}\right)}{1+b\left(\frac{\theta_1}{\mu_1}\right)}-\theta_1+\mu_2\left(\frac{\theta_1}{\mu_1}\right)\left(\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}\right)-\eta\left(\frac{r(\mu_1 K-\theta_1)}{\mu_1 K\beta_1}\right)$, $\lambda_2,\lambda_3 = \frac{T_2\pm\sqrt{T_2^2-4aD_2}}{2a} = \frac{(B_{11}+B_{22})\pm\sqrt{(B_{11}+B_{22})^2-4(B_{11}B_{22}-B_{12}B_{21})}}{2}$. ■

Theorem 5. *The equilibrium point stability $E_5(x^*,y^*,z^*)$ asymptotically stable if it satisfies the condition $a_1>0$, $a_3>0, a_3>0$ and $a_1a_2-a_3>0$.*

Proof. Substituting $E_5(x^*,y^*,z^*)$ into the Jacobian in eq. (3) yields.

$$J_{E_5} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix},$$

with each of its components being,

$$C_{11} = r(1-\frac{x^*}{K})-\frac{rx^*}{K}-\beta_1 y^*-\frac{\beta_2 z^*}{1+bx^*}+\frac{\beta_2 x^* z^* b}{(1+bx^*)^2},$$
$$C_{12} = \frac{\mu_1 y^*}{(1+\alpha z^*)},$$
$$C_{13} = \frac{\mu_2 z^*}{1+bx^*}-\frac{\mu_2 x^* z^* b}{(1+bx^*)^2}+\mu_3 y^* z^*,$$
$$C_{21} = -\beta_1 x^*,$$
$$C_{22} = \frac{\mu_1 x^*}{1+\alpha z^*}-\theta_1,$$
$$C_{23} = \mu_3 x^* z^*-\eta z^*,$$
$$C_{31} = -\frac{\beta_2 x^*}{1+bx^*},$$
$$C_{32} = -\frac{\mu_1 x^* y^* \alpha}{(1+\alpha z^*)^2},$$

$$C_{33} = \frac{\mu_2 x^*}{1+bx^*} - \theta_2 + \mu_3 x^* y^* - \eta y^*.$$

The eigenvalues are found by the characteristic equation $\det(J_{E_5} - \lambda I = 0)$.

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

with each of its components being,

$$\begin{aligned}
a_1 &= -(C_{11} + C_{22} + C_{33}),\\
a_2 &= C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33} - C_{23}C_{32} - C_{21}C_{12} - C_{31}C_{13},\\
a_3 &= -C_{11}C_{22}C_{33} - C_{21}C_{13}C_{32} - C_{31}C_{12}C_{23} + C_{11}C_{23}C_{32} + C_{21}C_{12}C_{33} + C_{31}C_{13}C_{22}.
\end{aligned}$$

Based on the Routh-Hurwitz stability condition, it states that the system is asymptotically stable under the condition $a_1 > 0, a_3 > 0$ and $a_1a_2 - a_3 > 0$. ■

3.3. Numerical Result

Numerical simulations were performed to confirm the agreement with the results of the analytical calculations. The simulation of the model is described based on several parameters presented in Table 1.

Table 1. Parameter value

parameters	Description	Value	Source
r	Intrinsic growth rate of prey	15	Assumption
K	Environmental carrying capacity	10	[25]
β_1	Predation rate of prey by the first predator	1.5	Assumption
β_2	Predation rate of prey by the second predator	0.9	Assumption
b	Handling time	0.5	[26]
μ_1	Conversion of prey biomass to first predator biomass	0.3*	[27]
μ_2	Conversion of prey biomass to second predator biomass	0.12*	Assumption
μ_3	Prey biomass conversion with kleptoparasitism success	0.1	Assumption
α	Effect of kleptoparasitism	0.12	Assumption
θ_1	Natural mortality rate of the first predator	0.3	Assumption
θ_2	Natural mortality rate of the second predator	0.2	[23]
η	anti-predator behaviour	0.1	[23]

**The parameters μ_1 and μ_2 are varied in the continuation simulation to analysis the behaviour at the equilibrium point.*

Numerical simulations are carried out by analysing the suitability of calculations with simulation results on Python software. The author performs numerical simulations by varying the conversion parameter of prey biomass into the first predator biomass μ_1 and varying the conversion parameter of prey biomass into the second predator biomass μ_2, which aims to determine the stability of the system solution at different values of μ_1. The selection of the parameter value μ_1 is based on analytical calculations with $\mu_1 > \frac{\theta_1}{K}$ or $\mu_1 < \frac{\theta_1}{K}$ and $\mu_2 > \theta_2(b + \frac{1}{K})$ or $\mu_2 < \theta_2(b + \frac{1}{K})$. By using the parameter value Table 1 which is substituted into the system of eq. (2) with the help of Maple 2021 software.

Case 1: parameter values $\mu_1 = 0.3$ and $\mu_2 = 0.12$

When $\mu_1 = 0.3$ and $\mu_1 = 0.12$ are based on the condition $\mu_1 > 0.03$ and $\mu_2 = 0.12$ are based on the condition $\mu_2 < 0.13$. Stability analysis with values $\mu_1 = 0.3$ and $\mu_1 = 0.12$ shows that there are 5 equilibrium points that exist, namely E_1, E_2, E_4, E_5, E_6, while the equilibrium points E_3 and E_7 are not equilibrated because they are negative. Stability analysis with the parameter values, the eigenvalues of each equilibrium point are obtained as follows.

1. $E_1 = (0,0,0)$ with $\lambda_1 = 15 > 0$, $\lambda_2 = -0.3 < 0$, and $\lambda_3 = -0.2$. The equilibrium point E_1 is saddle unstable.

2. $E_2 = (10,0,0)$ with $\lambda_1 = -15 < 0$, $\lambda_2 = 2.7 > 0$, and $\lambda_3 = -0.01538 < 0$. The equilibrium point E_2 is saddle unstable.
3. $E_4 = (1,9,0)$ with $\lambda_1 = -0.12258 < 0$, $\lambda_2 = -0.75 + 1.86748i < 0$, and $\lambda_3 = -0.75 + 1.86748i < 0$. The equilibrium point E_4 is spiral stable.
4. $E_5 = (1.13748, 8.4396, 1.14571)$ with $\lambda_1 = 0.094988 > 0$, $\lambda_2 = -0.77855 + 1.9963i < 0$, and $\lambda_3 = -0.77855 - 1.9963i < 0$. The equilibrium point E_5 is spiral unstable.
5. $E_6 = (4.7157, 0.11446, 30.9641)$ with $\lambda_1 = -0.19132 < 0$, $\lambda_2 = -0.64269 + 0.40121i < 0$, and $\lambda_3 = -0.64269 - 0.40121i < 0$. The equilibrium point E_6 is spiral stable.

There are two stable equilibrium points at $E_4 = (1,9,0)$ and $E_6 = (4.7157, 0.11446, 30.96415)$. Furthermore, the simulation results are illustrated with the parameter values in Table 1 by using the parameter value of prey biomass conversion to first predator biomass $\mu_1 = 0.3$ and using the parameter value of prey biomass conversion to second predator biomass $\mu_2 = 0.12$. The following is the result of the first time series graph when $\mu = 0.3$ with initial values $(0.1, 9.9, 0.4)$ and $(10, 0.1, 26)$.

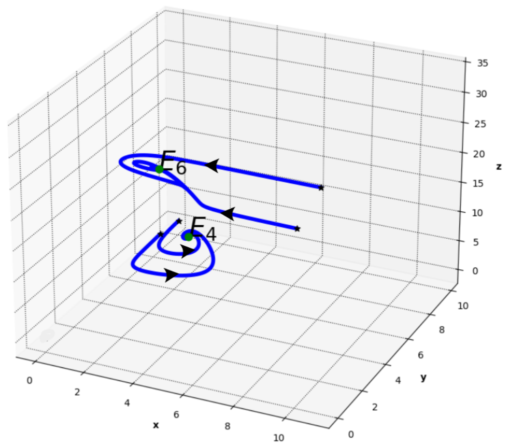
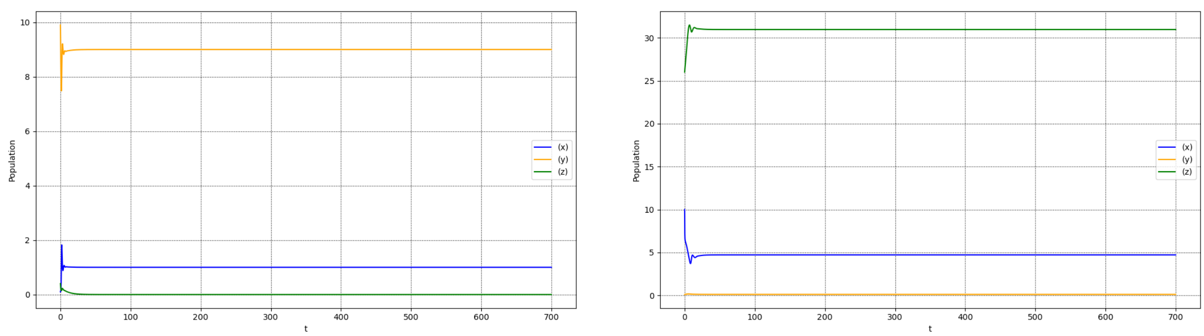


Figure 1. Phase Portrait at $\mu_1 = 0.3$

Figure 1 is a phase portrait that shows stability at two points, namely $E_4 = (1,9,0)$ which means that the prey population, the first predator population can coexist while the second predator population experiences extinction and $E_6 = (4.715, 0.114, 30.964)$ which means that the prey population, the first predator population, and the second predator population can coexist.



(a) **(b)**

Figure 2. (a) Time series at $\mu_1 = 0.3$ with initial values $(0.1, 9.9, 0.4)$. (b) Time series at $\mu_1 = 0.3$ with initial values $(10, 0.1, 26)$

Figure 2a shows that when the first prey biomass to predator biomass conversion parameter is $\mu_1 = 0.3$ with an initial value of $(0.1, 9.9, 0.4)$ it will stabilize towards the equilibrium point $E_4 = (1,9,0)$.

With an initial prey value of 0.1 to the equilibrium point $x = 1$. The first predator population with an initial value of 9.9 goes to the equilibrium point $y = 9$. With an initial value of 0.4, the second predator population goes to the equilibrium point $z = 0$ or extinction. Based on the time series in Figure 2a, the conversion parameter of prey biomass to first predator biomass of $\mu_1 = 0.3$ with initial values $(0.1, 9.9, 0.4)$ indicates that the prey population and the first predator can coexist, while the second predator population cannot coexist.

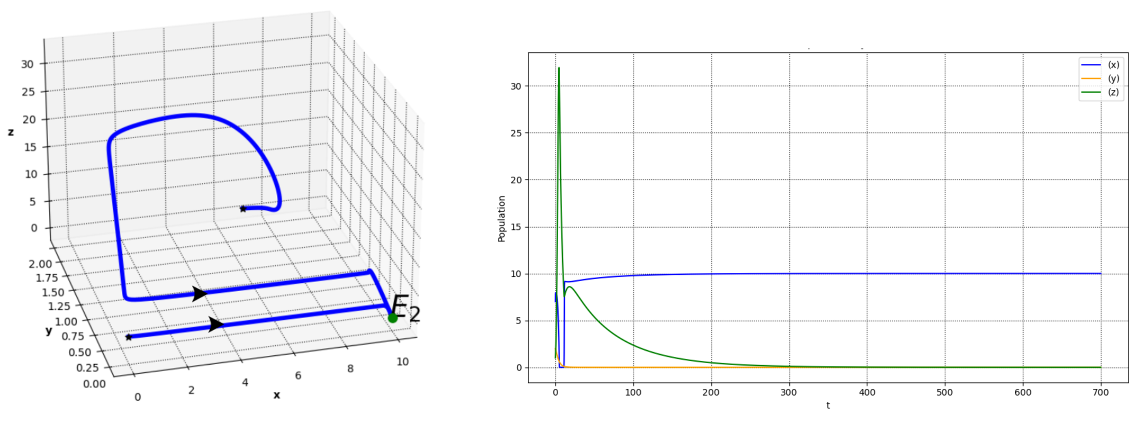
Figure 2b shows that when the first prey to predator conversion parameter is $\mu_1 = 0.3$ with initial values $(10, 0.1, 26)$, the system will stabilize towards the equilibrium point $E_6(4.7157, 0.11446, 30.96415)$. With an initial prey value of 10, the population goes to the equilibrium point $x = 4.7157$. The first predator population with an initial value of 0.1 goes to the equilibrium point $y = 0.11446$. The second predator population, With an initial value of 26, goes to the equilibrium point $z = 30.96415$. Based on the time series in Figure 2b, it shows that the second predator population has a larger population, 30.96415, compared to the first predator population of 0.11446, which means that when the first predator population is less, the parameter value μ_1 has less effect. Figure 2b also shows that the prey, first predator, and second predator populations can coexist.

Case 2: μ_1^* parameter values $\mu_1 = 0.02$

When parameter $\mu_1 = 0.02$ based on the requirement $\mu_1 < 0.03$. Stability analysis with values $\mu_1 = 0.02$ shows that there are 2 equilibrium points that exist, namely E_1 and E_2, while the equilibrium points E_3, E_4, E_5, E_6, E_7 are not equilibrated because they are negative. Stability analysis with the parameter values, the eigenvalues of each equilibrium point are obtained as follows.

1. $E_1 = (0, 0, 0)$ with $\lambda_1 = 15 > 0$, $\lambda_2 = -0.3 < 0$, and $\lambda_3 = -0.2$. The equilibrium point E_1 is saddle unstable.
2. $E_2 = (10, 0, 0)$ with $\lambda_1 = -15 < 0$, $\lambda_2 = -0.1 > 0$, and $\lambda_3 = -0.0154 < 0$. The equilibrium point E_2 is Noda stable.

There is one stable equilibrium point at equilibrium point $E_2 = (10, 0, 0)$. Furthermore, the simulation results are illustrated with the parameter values in Table 1 using the first prey biomass to predator biomass conversion parameter value $\mu_1^* = 0.02$.



(a) (b)

Figure 3. (a) Phase Portrait at $\mu_1 = 0.02$. (b) Time series at $\mu_1 = 0.02$.

Figure 3a is a phase portrait with initial values $(7, 2, 1)$ that shows stability at one point E_2 which is asymptotically stable with node type. The stability at the point $E_2 = (10, 0, 0)$ means that only the prey population can survive, while the population of the two predators is absent or extinct. The following will show the time series graph with the initial value $(7, 2, 1)$ with the corresponding parameters in Table Table 1 and when the conversion rate of prey biomass to the first predator biomass $\mu_1 = 0.02$.

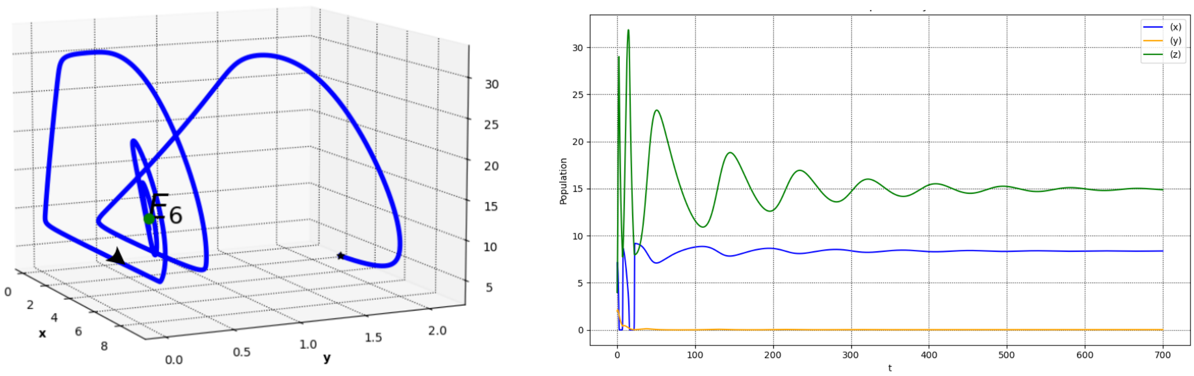
Figure 3b shows when the conversion rate of prey biomass to first predator biomass is $\mu_1 = 0.02$ with initial value $(7,2,1)$. With an initial prey value of 7 heads towards the equilibrium point $x = 10$. The population of the first predator and the second predator with initial values of 2 and 1 heads towards the equilibrium point $y, z = 0$. Based on the time series in Figure 3b shows that the prey population can survive or grow, the population of the first predator and second predator does not exist or extinct.

Case 3: μ_1^* *parameter values* $\mu_1 = 0.1$

When parameter $\mu_1 = 0.1$ based on the requirement $\mu_1 > 0.03$. Stability analysis with values $\mu_1 = 0.1$ shows that there are 4 equilibrium points that exist, namely E_1, E_2, E_3, E_6 while the equilibrium points E_3, E_5 and E_7 are not equilibrated because they are negative. Stability analysis with the parameter values, the eigenvalues of each equilibrium point are obtained as follows.

1. $E_1 = (0,0,0)$ with $\lambda_1 = 15 > 0$, $\lambda_2 = -0.3 < 0$, and $\lambda_3 = -0.2$. The equilibrium point E_1 is saddle unstable.
2. $E_2 = (10,0,0)$ with $\lambda_1 = -15 < 0$, $\lambda_2 = 0.7 > 0$, and $\lambda_3 = -0.01538 < 0$. The equilibrium point E_2 is saddle unstable.
3. $E_4 = (3,7,0)$ with $\lambda_1 = 1.33584 > 0$, $\lambda_2 = -3.6329 < 0$, and $\lambda_3 = -0.86706 < 0$. The equilibrium point E_4 is saddle unstable.
4. $E_6 = (8.37408, 0.02812, 14.92801)$ with $\lambda_1 = -10.57833 < 0$, $\lambda_2 = -0.006823 + 0.07292i < 0$, and $\lambda_3 = -0.006823 - 0.07292i < 0$. The equilibrium point E_6 is spiral stable.

There is one stable equilibrium point at equilibrium point $E_6 = (8.37408, 0.02812, 14.92801)$. Furthermore, the simulation results are illustrated with the parameter values in Table 1 using the first prey biomass to predator biomass conversion parameter value $\mu_1 = 0.1$.



(a) **(b)**

Figure 4. (a) Phase Portrait at $\mu_1 = 0.1$. (b) Time series at $\mu_1 = 0.1$.

Figure 4a is a phase portrait with initial values $(4,2,4)$ which shows stability at one point, namely point E_6, which is stable with a spiral type. Stability at point $E_6 = (8.37408, 0.02812, 14.92801)$ means that the prey population, the first predator population, and the second predator population can live side by side. The following will show the time series graph with the initial value of $(4,2,4)$ with the corresponding parameters in Table 1 and when the conversion rate of prey biomass to first predator biomass $\mu_1 = 0.1$.

Figure 4b shows when the conversion rate of prey biomass to the first predator biomass is $\mu_1 = 0.1$ with an initial value $(4,2,4)$. With an initial prey value of 4 towards the equilibrium point $x = 8.37408$. The first predator population with an initial value of 2 goes to the equilibrium point

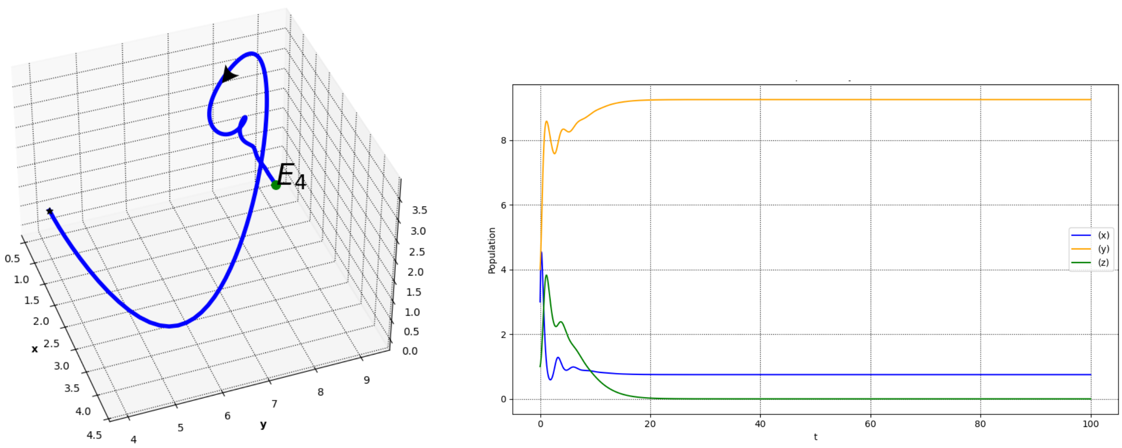
$y = 0.0281$. With an initial value of 4, the second predator goes to the equilibrium point $z = 14.9280$. Table 1 shows that the prey , first predator , and second predator populations can coexist.

Case 4: μ_1^* parameter values $\mu_1 = 0.4$

When parameter $\mu_1 = 0.4$ based on the requirement $\mu_1 > 0.03$. tability analysis with values $\mu_1 - 0.4$ shows that there are 5 equilibrium points that exist, namely E_1, E_2, E_4, E_5, E_6 while the equilibrium points E_4 and E_7 are not equilibrated because they are negative. Stability analysis with the parameter values, the eigenvalues of each equilibrium point are obtained as follows.

1. $E_1 = (0,0,0)$ with $\lambda_1 = 15 > 0$, $\lambda_2 = -0.3 < 0$, and $\lambda_3 = -0.2$. The equilibrium point E_1 is saddle unstable.
2. $E_2 = (10,0,0)$ with $\lambda_1 = -15 < 0$, $\lambda_2 = 3.7 > 0$, and $\lambda_3 = -0.01538$. The equilibrium point E_2 is saddle unstable.
3. $E_4 = (0.75, 9.25, 0)$ with $\lambda_1 = -0.36753 < 0$, $\lambda_2 = -0.5625 + 1.96114 < 0$, and $\lambda_3 = -0.5625 - 1.96114 < 0$. The equilibrium point E_1 is spiral stable.
4. $E_5 = (1.1604, 7.16948, 4.56002)$ with $\lambda_1 = 0.16288 > 0$, $\lambda_2 = -0.4637 + 2.3213i < 0$, and $\lambda_3 = -0.4637 - 2.3213i < 0$. The equilibrium point E_1 is spiral unstable.
5. $E_6 = (3.50335, 0.22515, 30.59281)$ with $\lambda_1 = -0.12012 > 0$, $\lambda_2 = 0.11831 + 1.22973i > 0$, and $\lambda_3 = 0.11831 - 1.22973i < 0$. The equilibrium point E_1 is spiral unstable.

There is one stable equilibrium point at equilibrium point $E4 = (0.75, 9.25, 0)$. Furthermore, the simulation results are illustrated with the parameter values in Table 1 using the first prey biomass to predator biomass conversion parameter value $\mu_1 = 0.4$.



(a) (b)

Figure 5. (a) Phase Portrait at $\mu_1 = 0.4$. (b) Time series at $\mu_1 = 0.4$.

Figure 5a is a phase portrait with initial values $(1,4,1)$ towards the equilibrium point $E4 = (0.75, 9.25, 0)$ so that it can be concluded that the stability of the equilibrium point $E4 = (0.75, 9.25, 0)$ is spiral stable. The stability at point $E4 = (0.75, 9.25, 0)$ shown in the phase portrait above means that the prey population and the first predator population remain while the second predator population is extinct.

Figure 5b shows that when the first prey biomass to predator biomass conversion parameter $\mu_1 = 0,4$ with initial value $(1,4,1)$ stabilizes towards the equilibrium point $E4 = (0.75, 9.25, 0)$. With an initial prey value of 1, the population stabilizes to the equilibrium point $x = 0.75$. The first predator population with an initial value of 4 goes to the equilibrium point $y = 9.25$. The second predator population with an initial value of 1 goes to the equilibrium point $z = 0$ or extinction. Based on the time series in Figure 5b, the first predator and prey populations have a larger population than the second predator population, which results in the second predator population experiencing extinction. The conversion rate of prey biomass to predator biomass of $\mu_1 = 0.4$ with initial population values (1, 4, 1) indicates that the first predator and prey populations can coexist, while the second predator population cannot coexist or is extinct.

Case 5: μ_2^* parameter values $\mu_2 = 0.158$

Parameter $\mu_2 = 0.158$ based on the requirement μ_2 based on the requirement $\mu_2 > 0.13$. Stability analysis with values $\mu_2 = 0.158$ shows that there are 5 equilibrium that exist, namely E_1, E_2, E_3, E_4, E_5, while the equilibrium points E_6 and E_7 are not equilibrated because they are negative. Stability analysis with the parameter values, the eigenvalues of each equilibrium point are obtained as follows.

1. $E_1 = (0,0,0)$ with $\lambda_1 = 15 > 0, \lambda_2 = -0.3 < 0$, and $E_3 = -0.2 < 0$. The equilibrium point E_1 is saddle unstable.
2. $E_2 = (10,0,0)$ with $\lambda_1 = -15 < 0, \lambda_2 = 2.7 > 0$ and $\lambda_3 = 0.043 > 0$ The equilibrium point E_2 is saddle unstable.
3. $E_3 = (4.167, 0, 32.002)$ with $\lambda_1 = -0.0981 < 0, \lambda_2 = -0.0791 + 0.72481i < 0$ and $\lambda_2 = -0.0791 - 0.72481i < 0$ The equilibrium point E_3 is spiral stable.
4. $E_4 = (1,9,0)$ with $\lambda_1 = -0.0981 < 0, \lambda_2 = -0.75 + 1.8674 < 0$ and $\lambda_3 = -0.75 - 1.8674 < 0$ The equilibrium point E_4 is spiral stable.
5. $E_5 = (1.106, 8.561, 0.8887)$ with $\lambda_1 = 0.0801 > 0, \lambda_2 = -0.7759 + 1.9703 < 0$ and $-0.7759 - 1.9703 < 0$ The equilibrium point E_5 is spiral unstable.

There are two stable equilibrium points at equilibrium points $E_3 = (4.167, 0, 32.002)$ and $E_4 = (1,9,0)$. Furthermore, the simulation results are illustrated with the parameter values in Table 1 by using the parameter value of prey biomass conversion to second predator biomass $\mu_2^* = 0.158$.

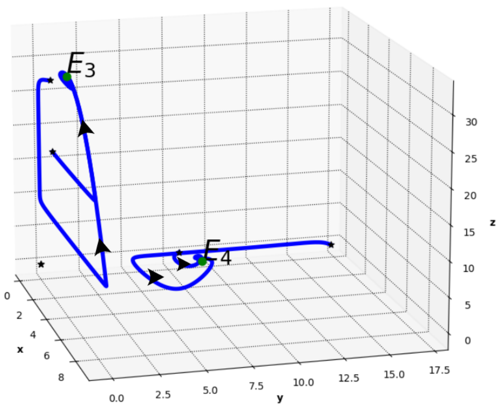
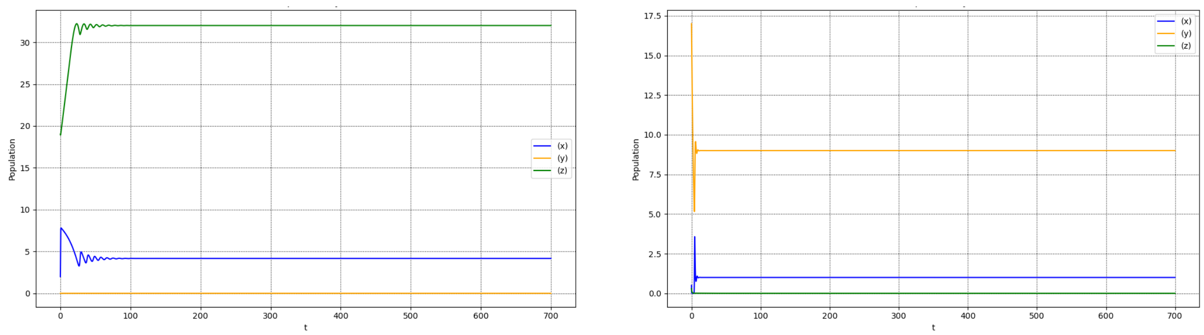


Figure 6. Phase Portrait at $\mu_1 = 0.158$

Figure 6 is a phase portrait that shows stability at two points namely $E_3 = (4.167, 0, 32.002)$ which means that the prey population, the second predator population can coexist while the first predator population population is extinct and $E_4 = (1,9,0)$ which means that the prey population, the first predator population can coexist and the second predator population is extinct. The following is the result of the first time series graph when $\mu_2^* = 0.158$ with initial values $(2, 0.0001, 19)$ and $(0.5, 17, 0.3)$.

Figure 7a shows that when the conversion parameter of prey biomass to second predator biomass is 0.158 with initial value $(2, 0.0001, 19)$ will be stable towards the equilibrium point $E_3 = (4.167, 0, 32.002)$. With an initial value of 2 prey population tails towards the equilibrium point $x = 4,167$. The first predator population with an initial value of 0.0001 heads towards the equilibrium point $y = 0$ or extinction. With an initial value of 19, the second predator goes to the equilibrium point $z = 32,002$. Based on



(a) **(b)**

Figure 7. (a) Time series at $\mu_2 = 0.158$ with initial values $(2, 0.0001, 19)$. (b) Time series at $\mu_1 = 0.158$ with initial values $(0.5, 17, 0.3)$.

the time series in Figure 7a, the conversion parameter of prey biomass to first predator biomass of $\mu_2^* = 0.158$ with initial values $(2, 0.0001, 19)$.

Figure 7b Shows that when the conversion parameter of prey biomass to second predator biomass is 0.158 with initial value $(0.5, 17, 0.3)$, the system will be stable towards the equilibrium point $E_4 = (1, 9, 0)$. The initial value of prey 0.5 goes to equilibrium point $x = 1$, the first predator of 17 goes to $y = 9$, and the second predator of 0,3 goes to equilibrium point $z = 0$ or extinction. Based on the time series in Figure 7b, the conversion parameter of prey biomass to first predator biomass of $\mu_2^* = 0.158$ with initial values $(0.5, 17, 0.3)$.

Case 6: μ_2^* parameter values $\mu_2 = 0.25$

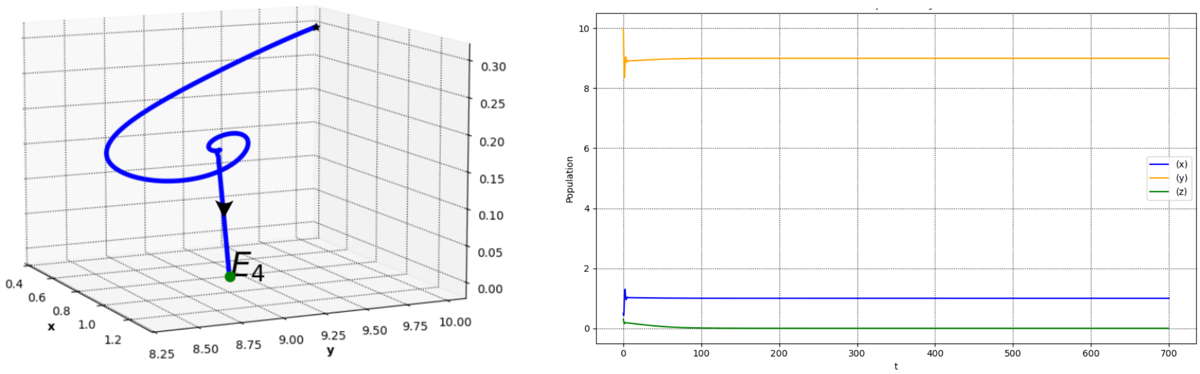
Parameter $\mu_2 = 0.25$ based on the requirement $\mu_2 > 0.25$. Stability analysis with values $\mu_2 = 0.2$ show that are 5 equilibrium points that exist, namely E_1, E_2, E_3, E_4, E_6 while the equilibrium points E_5 and E_7 are not equilibrated because they are negative. Stability analysis with the parameter values, the eigenvalues of each equilibrium point are obtained as follows.

1. $E_1 = (0, 0, 0)$ with $\lambda_1 = 15, \lambda_2 = -0.3$, and $\lambda_3 = -0.2$. The equilibrium point E_1 is saddle unsatable.
2. $E_2 = (10, 0, 0)$ with $\lambda_1 = -15 < 0, \lambda_2 = 2.7 > 0$, and $\lambda_3 = 0.1846 > 0$. The equilibrium point E_1 is saddle unsatable.
3. $E_3 = (1.4286, 0, 25.5102)$ with $\lambda_1 = 3.0407 > 0, \lambda_2 = 0.4735 > 0$, and $\lambda_3 = -0.1945 < 0$. The equilibrium point E_3 is saddle unsatable.
4. $E_4 = (1, 9, 0)$ with $\lambda_1 = -0.0387 < 0, \lambda_2 = -0.75 + 18674i < 0$, and $\lambda_3 = -0.75 + 18674i < 0$. The equilibrium point E_4 is spiral unsatable.
5. $E_6 = (1.039, 8.836, 0.327)$ with $\lambda_1 = 0.0358 > 0, \lambda_2 = -0.7633 + 1.9079i < 0$, and $\lambda_3 = -0.7633 - 1.9079i < 0$. The equilibrium point E_6 is spiral unstable

There is one stable equilibrium point at equilibrium point $E_4 = (1, 9, 0)$. Furthermore, the simulation results are illustrated with the parameter values in Table 1 sing the second prey biomass to predator biomass conversion parameter value $\mu_2 = 0.25$

Figure 8a is a phase potrait with initial values $(0.5, 10, 0.3)$. towards the equilibrium point $E_4 = (1, 9, 0)$ so that it can be concluded that the stability of the equilibrium $E_4 - (1, 9, 0)$ shown in the phase portrait above means that the prey population and the first predator population remain while the second predator population is extinct. The following will show the time series graph with initial values $(0.5, 10, 0.3)$ with parameters at the time of the conversion rate of prey biomass into the second predator biomass $\mu_2 = 0.25$

Figure 8b shows the rate of conversion of prey biomass into biomass of the second predator as $\mu_2 = 0.25$ with initial value $(0.5, 10, 0.3)$. With an initial prey value of 0.5 tails towards the equilibrium



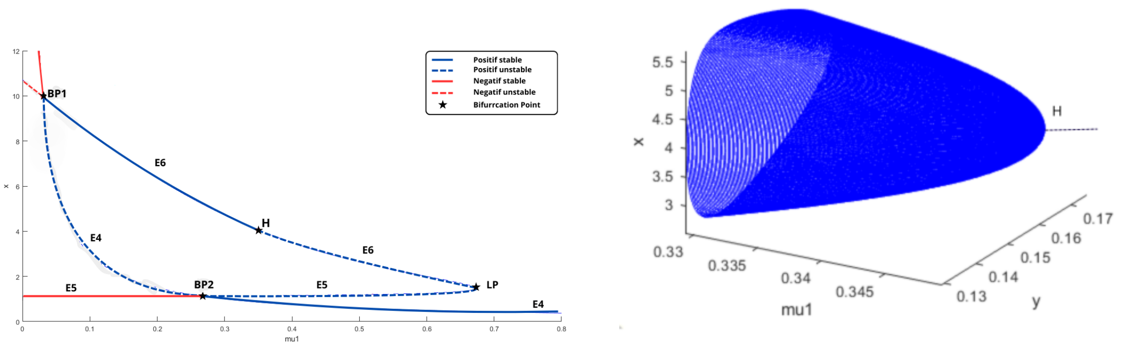
(a) **(b)**

Figure 8. (a) Phase Portrait at $\mu_2 = 0.25$. (b) Time series at $\mu_2 = 0.25$.

point $x = 1$. The first predator population with an initial value of 10 heads towards the equilibrium point $y = 9$. The first predator population with an initial value of 0.3 heads towards the equilibrium point $z = 0$ or extinction. The conversion rate of prey biomass to second predator biomass μ_2 performed by the first predator of 0.4 with initial population values $(1, 4, 1)$ indicates the first predator population (y) and prey (x) can coexist, while the second predator population (z) cannot coexist or become extinct.

3.4. Numerical Continuation with parameter values converting prey biomass into first predator biomass (μ_1)

Numerical continuation is performed on system eq. (2) by running the value of parameter μ_1, which is the conversion parameter of prey biomass into predator biomass. The results of numerical continuation of the parameter μ_1 cause changes in the stability of the equilibrium point illustrated by the bifurcation diagram as shown below.



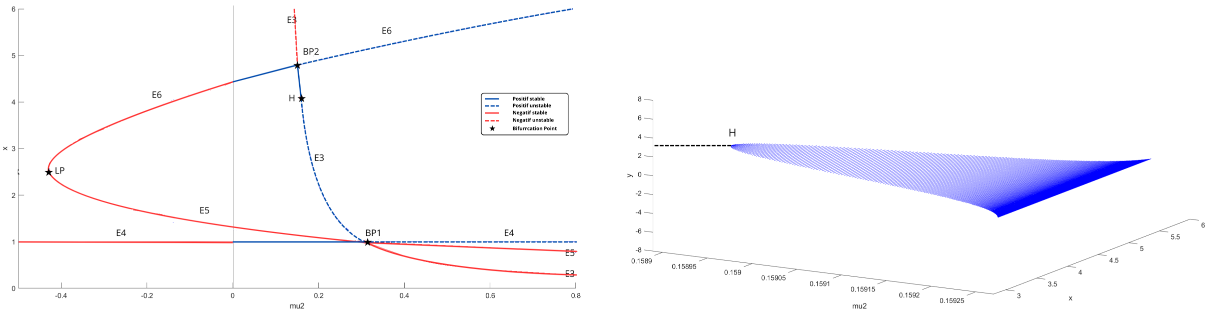
(a) **(b)**

Figure 9. (a) Bifurcation diagram μ_1. (b) Bifurcation diagram μ_1.

The numerical continuation results in the figure above show the overall change in the stability of the system due to changes in the value of the parameter μ_1. Continuation begins when the value of $\mu_1 = 0.3$ shows that there are 5 five equilibrium points that exist with two equilibrium points $E4$ and $E6$ stable or called bistable, $E1$, $E2$ and $E5$ are unstable. Then μ_1 is moved forward and BP1 (Branch Point) occurs at $\mu_1 = 0.265215$. The BP phenomenon is called Transcritcal Bifurcation which is characterised by the crossing of the two branches of the equilibrium duck in the bifurcation diagram at point E_4. This means that there is a change in stability from point E_4 which was originally stable to unstable when passing the value of the prey biomass conversion parameter to predator biomass with a parameter value of $\mu_1 > 0.265215$. Then BP2 occurs when $\mu_1 = 0.030052$ so that the interval $0.030052 < \mu_1 < 0.265215$ shows instability at point E4. After that, the continuation of μ_1 moves forward so that LP (Limit Point) occurs. This LP shows a Saddle Node bifurcation at $\mu_1 = 0.672858$ where two intersecting equilibrium points are E5 and E_6 in the interval $0.34942 < \mu_1 < 0.672858$. When μ_1 is moved backwards, a Hopf bifurcation (H) occurs at $\mu_1 = 0.349421$. Hopf bifurcation is a change that occurs in a system when a change in parameter value makes an initially stable equilibrium point unstable. When $\mu_1 < 0.349421$ the system changes from unstable to stable.

3.5. Numerical Continuation with parameter values converting prey biomass into second predator biomass (μ_2)

Numerical continuation is performed on the system eq. (2) by running the parameter value μ_2 which is the conversion parameter of prey biomass into predator biomass. The results of numerical continuation of the parameter μ_2 cause changes in the stability of the equilibrium point illustrated by the bifurcation diagram as shown below.



(a) **(b)**

Figure 10. (a) Bifurcation diagram μ_2. (b) Bifurcation diagram μ_2.

The numerical continuation results in the figure above show the overall change in the stability of the system due to changes in the value of the parameter μ_2. Continuation begins when the value of $\mu_2 = 0.12$ shows that there are 5 equilibrium points that exist with two equilibrium points E_4 and E_6 stable or called bistable. Then μ_2 is moved forward and BP1 (Branch Point) occurs at $\mu_2 = 0.31$. The BP phenomenon is called Transcritical Bifurcation which is characterised by the crossing of two branches of the equilibrium point in the bifurcation diagram, namely the point to E_4. This means that there is a change in stability from the point E_4 which was originally stable to unstable when passing the parameter value of the conversion of prey biomass to the second predator biomass with a parameter value of $\mu_2 > 0.31$. Then BP2 occurs when $\mu_2 = 0.151775$ so that when the parameter value.

4. Conclusion

The results of numerical simulations using phyton show that continuation or variation in the value of the first prey biomass to predator biomass conversion parameter μ_1 and variation in the value of the first prey biomass to predator biomass conversion parameter μ_2 affect the change of the system. Numerical simulation at the equilibrium point shows the same results as the analysis. When the value of the prey biomass conversion parameter into the first predator biomass is $\mu_1 = 0.3$ and $\mu_2 = 0.12$, it produces double stability or bistability where there is stability at two equilibrium points namely E_4 and E_6. When the parameter value μ_1 is varied while the value of $\mu_2 = 0.12$ produces several cases, namely when the parameter value $\mu_1 = 0.02$ produces one stable equilibrium point at point E_2, when the parameter $\mu_1 = 0.1$ produces one stable equilibrium point, namely E_6.Then when the parameter value of the conversion of prey biomass into the first predator biomass is $\mu_2 = 0.4$ produces one stable equilibrium point, namely E_4. Furthermore, when the parameter μ_2 is varied while $\mu_1 = 0.3$ produces several cases, namely when the parameter value $\mu_2 = 0.158$ produces two stable equilibrium points namely E_3 and E_4, and when the parameter value $\mu_2 = 0.25$ produces one stable equilibrium point namely E_4.

The results of this study indicate that adding kleptoparasitism and anti-predator behavior to the predator-prey model can provide a more realistic picture of the interactions between living things in nature. By combining two types of functional responses, namely Holling type I and type II, this model is able to explain how two predators with different hunting methods interact with one prey. This approach is not only useful for developing theories in the field of mathematical ecology, but can also serve as a starting point for understanding the population dynamics of wildlife in nature, especially in complex systems. In addition, this model can be used as a basis for further research, for example by adding spatial and stochastic aspects or real data from the field to make the results more applicable and closer to actual ecosystem conditions.

Supplementary Information

Author Contributions. **Tassha Aulia Putri:** Methodology, formal analysis, software, visualization, writing–original draft preparation.. **Dian Savitri:** Conceptualization, supervision, writing-review and editing.

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Conflict of interest. The authors declare that they have no conflicts of interest to report regarding the present study.

Data availability. The data supporting the findings of this study were obtained through numerical simulations and are available from the corresponding author upon reasonable request.

References

[1] Darmayani S, Hidana R, Latumahina FS, Nendissa SJ, Situmorang MV, Juniatmoko R, et al. Ekologi, Lingkungan Hidup dan Pembangunan. Paper Knowledge Toward a Media History of Documents. 2021:1-291.
[2] Din Q. Stability, bifurcation analysis and chaos control for a predator-prey system. JVC/Journal of Vibration and Control. 2019;25(3):612-26. doi:10.1177/1077546318790871.
[3] Sekar R, Sari FA. Interaksi Tak Terlihat Menguak Mutualisme, Komensalisme Dan Parasitisme Dalam Kehidupan Tumbuhan. Jurnal Sains Student Research (JSSR). 2025;3(4):1076-83. doi:https://doi.org/10.61722/jssr.v3i4.6070.
[4] Adrianto H. Buku Ajar Parasitologi Buku Pegangan Kuliah Mahasiswa Biologi Dan Pendidikan Biologi. Penerbit Andi; 2020.
[5] Padmakumar V, Shanthakumar M. The costs and benefits of kleptoparasitism in frigatebirds: An integrative review. International Journal of Forest, Animal And Fisheries Research. 2023;7(2):01-4. doi:10.22161/ijfaf.7.2.1.
[6] Wood KA, Stillman RA, Goss-Custard JD. The effect of kleptoparasite and host numbers on the risk of food-stealing in an avian assemblage. Journal of Avian Biology. 2015;46(6):589-96. doi:10.1111/jav.00705.
[7] Hadjichrysanthou C, Broom M, Rychtář J. Models of kleptoparasitism on networks: the effect of population structure on food stealing behaviour. Journal of Mathematical Biology. 2018;76(6):1465-88. doi:10.1007/s00285-017-1177-7.
[8] Morand-Ferron J, Sol D, Lefebvre L. Food stealing in birds: brain or brawn? Animal Behaviour. 2007;74(6):1725-34. doi:10.1016/j.anbehav.2007.04.031.
[9] Iyengar EV. Kleptoparasitic interactions throughout the animal kingdom and a re-evaluation, based on participant mobility, of the conditions promoting the evolution of kleptoparasitism. Biological Journal of the Linnean Society. 2008;93(4):745-62. doi:10.1111/j.1095-8312.2008.00954.x.
[10] Bhattacharjee D, Das D, Acharjee S, Dutta TK. Two predators, one prey model that integrates the effect of supplementary food resources due to one predator's kleptoparasitism under the possibility of retribution by the other predator. Heliyon. 2024;10(7). doi:10.1016/j.heliyon.2024.e28940.
[11] Carbone C, Frame L, Frame G, Malcolm J, Fanshawe J, FitzGibbon C, et al. Feeding success of African wild dogs (Lycaon pictus) in the Serengeti: The effects of group size and kleptoparasitism. Journal of Zoology. 2005;266(2):153-61. doi:10.1017/S0952836905006710.
[12] Källander H. Food hoarding and use of stored food by Rooks Corvus frugilegus. Bird Study. 2007;54(2):192-8. doi:10.1080/00063650709461475.

[13] Reader T. Strong interactions between species of phytophagous fly: A case of intraguild kleptoparasitism. Oikos. 2003;103(1):101-12. doi:10.1034/j.1600-0706.2003.12575.x.
[14] Martišová M, Bilde T, Pekár S. Sex-specific kleptoparasitic foraging in ant-eating spiders. Animal Behaviour. 2009;78(5):1115-8. doi:10.1016/j.anbehav.2009.07.025.
[15] Holling CS. Some Characteristics of Simple Types of Predation and Parasitism. The Canadian Entomologist. 1959;91(7):385-98. doi:10.4039/Ent91385-7.
[16] Mortoja SG, Panja P, Mondal SK. Dynamics of a predator-prey model with stage-structure on both species and anti-predator behavior. Informatics in Medicine Unlocked. 2018;10:50-7. doi:10.1016/j.imu.2017.12.004.
[17] Dawes JHP, Souza MO. A derivation of Holling's type I, II and III functional responses in predator-prey systems. Journal of Theoretical Biology. 2013;327:11-22. doi:10.1016/j.jtbi.2013.02.017.
[18] Goulart VDLR, Young RJ. Selfish behaviour as an antipredator response in schooling fish? Animal Behaviour. 2013;86(2):443-50. doi:10.1016/j.anbehav.2013.05.041.
[19] Holekamp KE, Smale L, Berg R, Cooper SM. Hunting rates and hunting success in the spotted hyena (Crocuta crocuta). Journal of Zoology. 1997;242(1):1-15. doi:10.1111/j.1469-7998.1997.tb02925.x.
[20] Smith JE, Memenis SK, Holekamp KE. Rank-related partner choice in the fission-fusion society of the spotted hyena (Crocuta crocuta). Behavioral Ecology and Sociobiology. 2007;61(5):753-65. doi:10.1007/s00265-006-0305-y.
[21] Höner OP, Wachter B, East ML, Hofer H. The response of spotted hyaenas to long-term changes in prey populations: Functional response and interspecific kleptoparasitism. Journal of Animal Ecology. 2002;71(2):236-46. doi:10.1046/j.1365-2656.2002.00596.x.
[22] Packer C, Scheel D, Pusey AE. Why lions form groups: food is not enough. American Naturalist. 1990;136(1):1-19. doi:10.1086/285079.
[23] Tang B, Xiao Y. Bifurcation analysis of a predator–prey model with anti-predator behaviour. Chaos, Solitons & Fractals. 2015 jan;70:58-68. doi:10.1016/j.chaos.2014.11.008.
[24] Mukherjee D. Stability and bifurcation of a two competing prey-one predator system with anti-predator behavior. Jambura Journal of Biomathematics. 2022 jun;3(1):1-11. doi:10.34312/jjbm.v3i1.13820.
[25] Savitri D, Panigoro H. Bifurkasi Hopf pada model prey-predator-super predator dengan fungsi respon yang berbeda. Jambura Journal of Biomathematics. 2020 dec;1(2):65-70. doi:10.34312/jjbm.v1i2.8399.
[26] Salsabila AN, Savitri D. Dynamical Analysis of Holling Tanner Prey Predators Model with Add Food in Second Level Predators. Jambura Journal of Biomathematics. 2024;5(2):63-70. doi:10.37905/jjbm.v5i2.25753.
[27] Focardi S, Materassi M, Innocenti G, Berzi D. Kleptoparasitism and scavenging can stabilize ecosystem dynamics. American Naturalist. 2017;190(3):398-409. doi:10.1086/692798.